

Springer correspondence for complex reflection groups

Toshiaki Shoji

ABSTRACT. This paper is a survey on the topics on the Springer correspondence related to the varieties such as the enhanced variety or the exotic symmetric space. We explain in the case of exotic symmetric space of higher level, the complex reflection group $S_n \ltimes (\mathbf{Z}/r\mathbf{Z})^n$ appears naturally in the framework of the Springer correspondence.

§1. Introduction

Springer correspondence is a canonical correspondence between the unipotent classes of a reductive group and the irreducible representations of its Weyl group, established by Springer [Sp] in 1976. In 1981, Lusztig [L1] found a way of reformulating Springer's theory in terms of the theory of perverse sheaves. In the same paper, he gave a geometric interpretation of Kostka polynomials in terms of the intersection cohomology associated to the unipotent classes in GL_n . In 1980's, Lusztig established the theory of character sheaves, developing the idea in [L1]. It became an interesting problem to generalize the theory of character sheaves to the case where the ambient variety is not a group. Recently, by [AH], [FGT], [K1] and [SS1-3], interesting examples, such as the enhanced variety and the exotic symmetric space, were found, which enjoy a satisfied theory of character sheaves and Springer correspondence, together with interesting relations with Kostka polynomials.

The exotic symmetric space and the enhanced variety of higher level are natural generalizations of those varieties mentioned above. In [S3], the theory of Springer correspondence for the exotic symmetric space of level r was established. Interestingly, the complex reflection group $S_n \ltimes (\mathbf{Z}/r\mathbf{Z})^n$ appears naturally in the description of the Springer correspondence. This paper is a survey on the Springer correspondence related to various varieties as above, based on the author's talk in the conference of Tsinghua Sanya International Mathematics Forum in December 2014.

S2. Springer correspondence for reductive groups

2.1. First we review some historical results on the Springer correspondence. The Springer correspondence is a natural correspondence between unipotent classes of a connected reductive group and irreducible representations of the associated Weyl group, first established by Springer [Sp]. Here we give a formulation due to Lusztig [L1] and Borho-MacPherson [BM] based on the theory of perverse sheaves.

Let G be a connected reductive group over an algebraically closed field \mathbf{k} . Let B be a Borel subgroup of G containing a maximal torus T of G , U the unipotent radical of B . Let $W = N_G(T)/T$ be the Weyl group of G . We denote by $\mathcal{B} = G/B$ the flag variety of G . Consider the morphism

$$\pi : \tilde{G} = \{(x, gB) \in G \times \mathcal{B} \mid g^{-1}xg \in B\} \rightarrow G$$

defined by $(x, gB) \mapsto x$. Then \tilde{G} is a smooth, irreducible variety, and π is a proper map. Let G_{reg} be the set of regular semisimple elements in G , and put $T_{\text{reg}} = T \cap G_{\text{reg}}$. Put $\tilde{G}_{\text{reg}} = \pi^{-1}(G_{\text{reg}})$ and let π_0 be the restriction of π on \tilde{G}_{reg} . Here $T_{\text{reg}} \times G/T \xrightarrow{\sim} \tilde{G}_{\text{reg}}$ by $(x, gT) \mapsto (gxg^{-1}, gB)$, and W acts on \tilde{G}_{reg} via the action $w : (x, gT) \mapsto (wxw^{-1}, gw^{-1}T)$ on $T_{\text{reg}} \times G/T$. Then $\pi_0 : \tilde{G}_{\text{reg}} \rightarrow G_{\text{reg}}$ turns out to be a Galois covering with Galois group W .

Consider a constant sheaf $\bar{\mathbf{Q}}_l$ on G_{reg} . Then the direct image $(\pi_0)_*\bar{\mathbf{Q}}_l$ is a semisimple local system, such that $\text{End}((\pi_0)_*\bar{\mathbf{Q}}_l) \simeq \bar{\mathbf{Q}}_l[W]$, and is decomposed along the irreducible representations of W ,

$$(2.1.1) \quad (\pi_0)_*\bar{\mathbf{Q}}_l \simeq \bigoplus_{\rho \in W^\wedge} \rho \otimes \mathcal{L}_\rho,$$

where W^\wedge denotes the set of irreducible representations of W over $\bar{\mathbf{Q}}_l$, up to isomorphism, and $\mathcal{L}_\rho = \text{Hom}_W((\pi_0)_*\bar{\mathbf{Q}}_l, \rho)$ is an irreducible local system on G_{reg} .

Since G_{reg} is an open dense smooth subset of G , one can consider the intersection cohomology complex $\text{IC}(G, (\pi_0)_*\bar{\mathbf{Q}}_l)$ on G . Let $\pi_*\bar{\mathbf{Q}}_l$ be the direct image of the constant sheaf $\bar{\mathbf{Q}}_l$ on \tilde{G} (an object in the derived category of $\bar{\mathbf{Q}}_l$ -sheaves). It is proved by Lusztig [L1] that $\pi_*\bar{\mathbf{Q}}_l \simeq \text{IC}(G, (\pi_0)_*\bar{\mathbf{Q}}_l)$. Hence $\pi_*\bar{\mathbf{Q}}_l[\dim G]$ is a semisimple perverse sheaf, equipped with W -action, and is decomposed as

$$(2.1.2) \quad \pi_*\bar{\mathbf{Q}}_l[\dim G] \simeq \bigoplus_{\rho \in W^\wedge} \rho \otimes \text{IC}(G, \mathcal{L}_\rho)[\dim G].$$

2.2. Let G_{uni} be the set of unipotent elements in G , which is a closed subvariety of G , and is called the **unipotent variety** of G . It is known that G_{uni} consists of finitely many G -orbits, under the conjugation action of G . Let \mathcal{N}_G be the set of all the pairs (C, \mathcal{E}) , where C is a unipotent class, and \mathcal{E} is a G -equivariant simple local system on C . If we fix $x \in C$, and let $A_G(x) = Z_G(x)/Z_G^0(x)$ be the component group of $Z_G(x)$. Then the set of

G -equivariant simple local system on C is in bijection with the set $A_G(x)^\wedge$ of irreducible representations of (the finite group) $A_G(x)$. Thus \mathcal{N}_G can be written as

$$(2.2.1) \quad \mathcal{N}_G \simeq \{(x, \eta) \mid x \in G_{\text{uni}}/\sim, \eta \in A_G(x)^\wedge\},$$

where x runs over a set of representatives of G -orbits in G_{uni} . Let \overline{C} be the closure of C in G_{uni} , and consider the intersection cohomology $\text{IC}(\overline{C}, \mathcal{E})$ on \overline{C} . By the extension by zero, we regard $A_{(C, \mathcal{E})} = \text{IC}(\overline{C}, \mathcal{E})[\dim C]$ as a perverse sheaf on G_{uni} . Then $A_{(C, \mathcal{E})}$ is an G -equivariant simple perverse sheaf on G_{uni} , and the set of (isomorphism class of) G -equivariant simple perverse sheaves on G_{uni} is given by $\{A_{(C, \mathcal{E})} \mid (C, \mathcal{E}) \in \mathcal{N}_G\}$.

Put $\tilde{G}_{\text{uni}} = \pi^{-1}(G_{\text{uni}})$, and let π_1 be the restriction of π on \tilde{G}_{uni} . Thus

$$\pi_1 : \tilde{G}_{\text{uni}} = \{(x, gB) \in G_{\text{uni}} \times \mathcal{B} \mid g^{-1}xg \in U\} \rightarrow G_{\text{uni}}$$

with $\pi_1(x, gB) = x$. By the base change theorem, the restriction of $\pi_* \bar{\mathbf{Q}}_l$ on G_{uni} is isomorphic to $(\pi_1)_* \bar{\mathbf{Q}}_l$, hence $(\pi_1)_* \bar{\mathbf{Q}}_l$ has a natural action of W . The following result is known as the Springer correspondence.

Theorem 2.3 (Borho-MacPherson [BM]). *Let the notations be as above.*

- (i) $(\pi_1)_* \bar{\mathbf{Q}}_l[\dim G_{\text{uni}}]$ is a semisimple perverse sheaf on G_{uni} , equipped with W -action, and is decomposed as

$$(\pi_1)_* \bar{\mathbf{Q}}_l[\dim G_{\text{uni}}] \simeq \bigoplus_{(C, \mathcal{E}) \in \mathcal{N}_G} V_{(C, \mathcal{E})} \otimes A_{(C, \mathcal{E})},$$

where $V_{(C, \mathcal{E})}$ is an irreducible representation of W if it is non-zero.

- (ii) For any $\rho \in W^\wedge$, there exists a unique pair $(C, \mathcal{E}) \in \mathcal{N}_G$ such that

$$\text{IC}(G, \mathcal{L}_\rho)|_{G_{\text{uni}}} \simeq \text{IC}(\overline{C}, \mathcal{E})[\dim C - \dim G_{\text{uni}}]$$

and that $\rho \simeq V_{(C, \mathcal{E})}$.

- (iii) The correspondence $\rho \mapsto (C, \mathcal{E})$ in (ii) gives a bijection

$$W^\wedge \simeq \{(C, \mathcal{E}) \in \mathcal{N}_G \mid V_{(C, \mathcal{E})} \neq 0\}.$$

2.4. For $x \in G$, put $\mathcal{B}_x = \{gB \in \mathcal{B} \mid g^{-1}xg \in B\}$. Then \mathcal{B}_x is a closed subvariety of \mathcal{B} . Since $\pi^{-1}(x) \simeq \mathcal{B}_x$, \mathcal{B}_x is called the **Springer fibre** of x . Since $\pi_* \bar{\mathbf{Q}}_l$ is a complex with W -action, the stalk $\mathcal{H}_x^i(\pi_* \bar{\mathbf{Q}}_l)$ at $x \in G$ of the cohomology sheaf $\mathcal{H}^i(\pi_* \bar{\mathbf{Q}}_l)$ has a natural action of W . By a

general theory $\mathcal{H}_x^i(\pi_* \bar{\mathbf{Q}}_l) \simeq H^i(\mathcal{B}_x, \bar{\mathbf{Q}}_l)$, so the cohomology group $H^i(\mathcal{B}_x, \bar{\mathbf{Q}}_l)$ (a finite dimensional vector space over $\bar{\mathbf{Q}}_l$) has a structure of W -module. The representation of W on $H^i(\mathcal{B}_x, \bar{\mathbf{Q}}_l)$ is called the **Springer representation** of W .

Note that $Z_G(x)$ acts on \mathcal{B}_x by the left multiplication, and it induces an action of $Z_G(x)$ on $H^i(\mathcal{B}_x, \bar{\mathbf{Q}}_l)$. Since $Z_G(x)$ acts trivially on $H^i(\mathcal{B}_x, \bar{\mathbf{Q}}_l)$, $A_G(x)$ acts on $H^i(\mathcal{B}_x, \bar{\mathbf{Q}}_l)$. It is known that this action of $A_G(x)$ on $H^i(\mathcal{B}_x, \bar{\mathbf{Q}}_l)$ commutes with the Springer action of W . Thus $H^i(\mathcal{B}_x, \bar{\mathbf{Q}}_l)$ has a structure of $W \times A_G(x)$ -module. Let $d_x = \dim \mathcal{B}_x$. Then it is known, for $x \in C$, that

$$(2.4.1) \quad d_x = \frac{1}{2}(\dim G_{\text{uni}} - \dim C).$$

We now concentrate on the top cohomolgy group $H^{2d_x}(\mathcal{B}_x, \bar{\mathbf{Q}}_l)$ as a $W \times A_G(x)$ -module. Theorem 2.3 can be reformulated in terms of the Springer representations of W , which is the original form of the Springer correspondence due to Spriner [Sp].

Corollary 2.5 (Springer [Sp]). *Write the decomposition of $W \times A_G(x)$ -module $H^{2d_x}(\mathcal{B}_x, \bar{\mathbf{Q}}_l)$ as follows;*

$$H^{2d_x}(\mathcal{B}_x, \bar{\mathbf{Q}}_l) \simeq \bigoplus_{\eta \in A_G(x)^\wedge} V_{(x,\eta)} \otimes \eta.$$

Then $V_{(x,\eta)}$ is an irreducible W -module if it is non-zero. Any irreducible representation of W can be realized as $V_{(x,\eta)}$ for a unique pair $(x,\eta) \in \mathcal{N}_G$ (so $V_{(x,\eta)}$ coincides with $V_{(C,\mathcal{E})}$ if $(x,\eta) \leftrightarrow (C,\mathcal{E})$ in (2.2.1)).

§3. Kostka polynomials

3.1. We consider the Springer correspondence in Theorem 2.3 and Corollary 2.5 in the special case where $G = GL_n = GL(V)$. In this case, the set of unipotent classes in G_{uni} is parametrized by the set \mathcal{P}_n of partitions of n , via Jorpdan normal form. We denote by C_λ the class corresponding to $\lambda \in \mathcal{P}_n$. In the case of GL_n , it is known that $Z_G(x)$ is connected, and so $A_G(x) = \{1\}$ for any $x \in G$. Thus $\mathcal{N}_G = \{(C, \bar{\mathbf{Q}}_l) \mid \bar{\mathbf{Q}}_l : \text{constant sheaf on } C\}$, and \mathcal{N}_G can be identified with the set G_{uni}/\sim of unipotent classes in G . Moreover, $W \simeq S_n$, the symmetric group of degree n , and the Springer correspondence gives a bijection $S_n^\wedge \simeq G_{\text{uni}}/\sim$. It is known that irreducible representations of S_n are naturally parametrized by \mathcal{P}_n . We denote by V_λ the irreducible representation of S_n corresponding to λ (here we use the labelling such that $V_\lambda = 1_{S_n}$ if $\lambda = (n)$, and V_λ is the sign representation if $\lambda = (1^n)$.) Then it

can be verified that the Springer correspondence is actually given by $V_\lambda \leftrightarrow C_\lambda$. In this case, the formula in Theorem 2.3 (i) is written as

$$(3.1.1) \quad (\pi_1)_* \bar{\mathbf{Q}}_l[\dim G_{\text{uni}}] \simeq \bigoplus_{\lambda \in \mathcal{P}_n} V_\lambda \otimes \text{IC}(\bar{C}_\lambda, \bar{\mathbf{Q}}_l)[\dim C_\lambda]$$

Moreover, Corollary 2.5 implies, for $x \in C_\lambda$, that $H^{2d_x}(\mathcal{B}_x, \bar{\mathbf{Q}}_l) \simeq V_\lambda$ as S_n -modules, and any irreducible representation of S_n can be realized by such a Springer module.

Recall that the partial order, called the **dominance order**, on \mathcal{P}_n is defined as follows; for $\lambda, \mu \in \mathcal{P}_n$, write $\lambda = (\lambda_1, \dots, \lambda_k), \mu = (\mu_1, \dots, \mu_k) \in \mathcal{P}_n$ for some common k (by allowing 0 for the parts λ_i, μ_i). Then $\mu \leq \lambda$ if $\sum_{i=1}^j \mu_i \leq \sum_{i=1}^j \lambda_i$ for each j .

By using the dominance order, the closure \bar{C}_λ can be described as follows;

$$(3.1.2) \quad \bar{C}_\lambda = \coprod_{\mu \leq \lambda} C_\mu.$$

3.2. Kostka polynomials $K_{\lambda, \mu}(t) \in \mathbf{Z}[t]$ are well-known polynomials in the combinatorial theory, indexed by partitions $\lambda, \mu \in \mathcal{P}_n$. They are given as the coefficients of the transition matrix between the basis of Schur functions and that of Hall-Littlewood functions in the space of symmetric functions.

For $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathcal{P}_n$, we define an integer $n(\lambda) \in \mathbf{Z}_{\geq 0}$ by $n(\lambda) = \sum_{i=1}^k (i-1)\lambda_i$. Then it is known that $K_{\lambda, \mu}(t)$ is a monic of degree $n(\mu) - n(\lambda)$ if $\mu \leq \lambda$, and $K_{\lambda, \mu} = 0$ otherwise. We define a modified Kostka polynomial $\tilde{K}_{\lambda, \mu}(t)$ by $\tilde{K}_{\lambda, \mu}(t) = t^{n(\mu)} K_{\lambda, \mu}(t^{-1})$.

The following interesting formula was proved by Lusztig in 1981.

Theorem 3.3 (Lusztig[L1]). *For $\lambda \in \mathcal{P}_n$, consider the intersection cohomology $K = \text{IC}(\bar{C}_\lambda, \bar{\mathbf{Q}}_l)$ on G_{uni} . Let $\mathcal{H}_x^i K$ be the stalk at $x \in G_{\text{uni}}$ of the cohomology sheaf $\mathcal{H}^i K$.*

- (i) $\mathcal{H}^i K = 0$ for odd i .
- (ii) Assume that $\mu \leq \lambda$, and take $x \in C_\mu \subset \bar{C}_\lambda$. Then

$$(3.3.1) \quad \tilde{K}_{\lambda, \mu}(t) = t^{n(\lambda)} \sum_{i \geq 0} (\dim \mathcal{H}_x^{2i} K) t^i.$$

Remarks 3.4. (i) Lascoux-Schützenberger theorem ([M, III, (6.5)]) gives a combinatorial description of Kostka polynomials as follows;

$$(3.4.1) \quad K_{\lambda, \mu}(t) = \sum_{T \in SST(\lambda, \mu)} t^{c(T)},$$

where $SST(\lambda, \mu)$ is the set of semistandard tableaux of shape λ and weight μ , and $c(T)$ is the charge of the tableau T (see [M] for the definition). This theorem implies that the coefficients of $K_{\lambda, \mu}(t)$ are non-negative integers. Theorem 3.3 gives an alternate proof of this fact.

(ii) Assume that \mathbf{k} is an algebraic closure of a finite field \mathbf{F}_q , and consider a finite subgroup $GL_n(\mathbf{F}_q)$ of GL_n . It was known that modified Kostka polynomials $\tilde{K}_{\lambda, \mu}(t)$ evaluated at $t = q$ give some character values of certain irreducible representations of $GL_n(\mathbf{F}_q)$ over \mathbf{Q}_l . Then (3.3.1) is regarded as a formula which describes some character values of $GL_n(\mathbf{F}_q)$ in terms of the intersection cohomology associated to G -orbits in G . This point of view was later generalized extensively by Lusztig, and he established the theory of character sheaves ([L3]) on connected reductive groups G , which is a geometric theory describing all the character values of $G(\mathbf{F}_q)$ in terms of certain intersection cohomology associated to G . The theory of Springer correspondence was later generalized by Lusztig to the theory of generalized Springer correspondence ([L2]), which plays an essential role in the theory of character sheaves.

§4. Enhanced variety \mathcal{X}^{en}

4.1. It is an interesting problem to generalize the theory of character sheaves to the case where the ambient variety is not a connected reductive group. In fact, in [L4], Lusztig developed the theory of character sheaves on disconnected reductive groups. Boyarchenko and Drinfeld [BD] developed the theory of character sheaves on unipotent groups in positive characteristic. It is also interesting to replace G by a variety \mathcal{X} on which G acts. In fact, Ginzburg [Gi] defined the character sheaves on the symmetric space G/H (H is a closed subgroup of G). Recently, some other examples of \mathcal{X} , such as the enhanced variety and the exotic symmetric space, were found that they enjoy a satisfied theory of character sheaves.

4.2. Before going to the discussion on the enhanced variety and the exotic space, we prepare some general notion from the combinatorics. For a positive integer r , we denote by $\mathcal{P}_{n,r}$ the set of r -tuple of partitions $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ such that $\sum_{i=1}^r |\lambda^{(i)}| = n$. We shall define a partial order $\boldsymbol{\mu} \leq \boldsymbol{\lambda}$ in $\mathcal{P}_{n,r}$. For $\boldsymbol{\lambda} \in \mathcal{P}_{n,r}$, express the partition $\lambda^{(i)}$ as $\lambda^{(i)} = (\lambda_1^{(i)}, \dots, \lambda_k^{(i)})$ for some common k , by allowing zero on $\lambda_j^{(i)}$, and define a composition $c(\boldsymbol{\lambda})$ of n by

$$c(\boldsymbol{\lambda}) = (\lambda_1^{(1)}, \dots, \lambda_1^{(r)}, \lambda_2^{(1)}, \dots, \lambda_2^{(r)}, \dots, \lambda_k^{(1)}, \dots, \lambda_k^{(r)}).$$

Then define $\boldsymbol{\mu} \leq \boldsymbol{\lambda}$ by the condition $c(\boldsymbol{\mu}) \leq c(\boldsymbol{\lambda})$, by using the dominance order on \mathcal{P}_n (can be defined similarly for the set of compositions of n).

The n -function on \mathcal{P}_n , $\lambda \mapsto n(\lambda)$, in 3.2 is also generalized to the case of $\mathcal{P}_{n,r}$. We define a function $a : \mathcal{P}_{n,r} \rightarrow \mathbf{Z}_{\geq 0}$ as follows; for each $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$, put

$$(4.2.1) \quad a(\boldsymbol{\lambda}) = r \cdot n(\boldsymbol{\lambda}) + |\lambda^{(2)}| + 2|\lambda^{(3)}| + \dots + (r-1)|\lambda^{(r)}|,$$

where $n(\boldsymbol{\lambda}) = \sum_{i=1}^r n(\lambda^{(i)})$. In particular, in the case where $r = 2$, $a(\boldsymbol{\lambda}) = 2n(\boldsymbol{\lambda}) + |\lambda^{(2)}|$.

4.3. First we shall introduce the enhanced variety. Let $G = GL_n = GL(V)$ be as in §3, where V is an n -dimensional vector space over \mathbf{k} . We consider the direct product $\mathcal{X} = G \times V$, on which G acts by the conjugation action on the first factor, and the natural action on the second factor. The variety $\mathcal{X} = \mathcal{X}^{\text{en}}$ is called the **enhanced variety**. Put $\mathcal{X}_{\text{uni}} = G_{\text{uni}} \times V$, which is a G -stable closed subset of G . \mathcal{X}_{uni} has a role of the unipotent variety, and is called the unipotent variety of \mathcal{X} . The geometry of the enhanced varieties \mathcal{X} and \mathcal{X}_{uni} is studied extensively by Achar-Henderson [AH], and Finkelberg-Ginzburg-Travkin [FGT]. In particular, the theory of character sheaves and the Springer correspondence on \mathcal{X} were discussed in [FGT]. Note that \mathcal{X}_{uni} is isomorphic to the **enhanced nilpotent cone** $\mathfrak{g}_{\text{nil}} \times V$ (here $\mathfrak{g}_{\text{nil}}$ is the nilpotent cone of the Lie algebra \mathfrak{g} of G) introduced by [AH]. The following fact was found by [AH] and Travkin [T], independently.

Lemma 4.4 ([AH], [T]). *Let $\mathcal{X}_{\text{uni}}/\sim$ be the set of G -orbits in \mathcal{X}_{uni} . Then \mathcal{X}_{uni} is in bijection with $\mathcal{P}_{n,2}$.*

4.5. Following [AH], we shall give an explicit correspondence. Take $z = (x, v) \in G_{\text{uni}} \times V$. Put $E^x = \{y \in \text{End}(V) \mid xy = yx\}$. E^x is a subalgebra of $\text{End}(V)$ containing x . Then $V_z = E^x v$ is an x -stable subspace of x . Let λ' (resp. λ'') be the Jordan type of $x|_{V_z}$ (resp. $x|_{V/V_z}$). We have $\boldsymbol{\lambda} = (\lambda', \lambda'') \in \mathcal{P}_{n,2}$, and the assignment $(x, v) \mapsto \boldsymbol{\lambda}$ gives the required parametrization of G -orbits in \mathcal{X}_{uni} . We denote by $\mathcal{O}_{\boldsymbol{\lambda}}$ the G -orbit corresponding to $\boldsymbol{\lambda}$.

The closure relations for $\mathcal{O}_{\boldsymbol{\lambda}}$ were determined by [AH].

Lemma 4.6 ([AH, Thm. 3.9]). *For each $\boldsymbol{\lambda} \in \mathcal{P}_{n,2}$,*

$$\overline{\mathcal{O}_{\boldsymbol{\lambda}}} = \coprod_{\boldsymbol{\mu} \leq \boldsymbol{\lambda}} \mathcal{O}_{\boldsymbol{\mu}}.$$

4.7. Following [FGT], we shall describe the Springer correspondence for \mathcal{X}_{uni} . We follow the notation in §3. Let $(M_i)_{0 \leq i \leq n}$ be the total flag in V whose stabilizer in G equals to B . Thus we can choose a basis $\{e_1, \dots, e_n\}$ of V consisting of weight vectors for T such that $M_i = \langle e_1, \dots, e_i \rangle$. For an integer m such that $0 \leq m \leq n$, we define

$$\begin{aligned}\widetilde{\mathcal{X}}_m &= \{(x, v, gB) \in G \times V \times \mathcal{B} \mid g^{-1}xg \in B, g^{-1}v \in M_m\}, \\ \mathcal{X}_m &= \bigcup_{g \in G} g(B \times M_m),\end{aligned}$$

and define a map $\pi_m : \widetilde{\mathcal{X}}_m \rightarrow \mathcal{X}$ by $(x, v, gB) \mapsto (x, v)$. Then π_m is a surjective map onto \mathcal{X}_m . Since π_m is proper, \mathcal{X}_m is a closed subvariety of \mathcal{X} . We also consider their restriction on the unipotent variety,

$$\begin{aligned}\widetilde{\mathcal{X}}_{m, \text{uni}} &= \{(x, v, gB) \in G_{\text{uni}} \times V \times \mathcal{B} \mid g^{-1}xg \in U, g^{-1}v \in M_m\}, \\ \mathcal{X}_{m, \text{uni}} &= \bigcup_{g \in G} g(U \times M_m),\end{aligned}$$

and define a map $\pi_{1,m} : \mathcal{X}_{m, \text{uni}} \rightarrow \mathcal{X}_m$ by $(x, v, gB) \mapsto (x, v)$. Then $\pi_{1,m}$ is a proper surjective map onto $\mathcal{X}_{m, \text{uni}}$. Put $\mathbf{m} = (m, n - m)$, and let $S_{\mathbf{m}} = S_m \times S_{n-m}$ be the Weyl subgroup of S_n , which is the stabilizer of M_m in S_n (under the embedding of S_n into G as a permutation group of $\{e_1, \dots, e_n\}$). Put

$$M_m^0 = \{v \in M_m \mid v = \sum_{i=1}^m a_i e_i \text{ with } a_i \neq 0 \text{ for all } i\},$$

and put $\mathcal{Y}_m^0 = \bigcup_{g \in G} g(T_{\text{reg}} \times M_m^0)$. Then \mathcal{Y}_m^0 is an open dense subset of \mathcal{X}_m . Put $d_m = \dim \mathcal{X}_m$. It is proved by Finkelberg and Ginzburg [FG, Cor. 5.4.2] that $(\pi_m)_* \bar{\mathbf{Q}}_l[d_m]$ is a semisimple perverse sheaf on \mathcal{X}_m , equipped with $S_{\mathbf{m}}$ -action, and is decomposed as

$$(4.7.1) \quad (\pi_m)_* \bar{\mathbf{Q}}_l[d_m] \simeq \bigoplus_{\rho \in S_{\mathbf{m}}^\wedge} \rho \otimes \text{IC}(\mathcal{X}_m, \mathcal{L}_\rho)[d_m],$$

where \mathcal{L}_ρ is a simple local system on \mathcal{Y}_m^0 .

Let $\mathcal{P}(\mathbf{m})$ be the set of $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}) \in \mathcal{P}_{n,2}$ such that $|\lambda^{(1)}| = m, |\lambda^{(2)}| = n - m$. It is clear that irreducible representations of $S_{\mathbf{m}} = S_m \times S_{n-m}$ are parametrized by $\mathcal{P}(\mathbf{m})$. We denote by $V_{\boldsymbol{\lambda}}$ the irreducible representation of $S_{\mathbf{m}}$ corresponding to $\boldsymbol{\lambda}$.

The following result was proved in [SS3, Thm. 2.11] (see also the proof of [FGT, Thm. 1]), which is regarded as an analogue of the Springer correspondence for the case \mathcal{X}_{uni} .

Theorem 4.8 ([SS3]). *Put $d'_m = \dim \mathcal{X}_{m,\text{uni}}$.*

- (i) *$(\pi_{1,m})_* \bar{\mathbf{Q}}_l[d'_m]$ is a semisimple perverse sheaf on $\mathcal{X}_{m,\text{uni}}$, equipped with $S_{\mathbf{m}}$ -action, and is decomposed as*

$$(\pi_{1,m})_* \bar{\mathbf{Q}}_l[d'_m] \simeq \bigoplus_{\lambda \in \mathcal{P}(\mathbf{m})} V_{\lambda} \otimes \text{IC}(\bar{\mathcal{O}}_{\lambda}, \bar{\mathbf{Q}}_l)[\dim \mathcal{O}_{\lambda}].$$

- (ii) *For each $\lambda \in \mathcal{P}(\mathbf{m})$, let \mathcal{L}_{ρ} be the simple local system on \mathcal{Y}_m^0 corresponding to $\rho = V_{\lambda} \in S_{\mathbf{m}}^{\wedge}$. Then we have*

$$\text{IC}(\mathcal{X}_m, \mathcal{L}_{\rho})|_{\mathcal{X}_{m,\text{uni}}} \simeq \text{IC}(\bar{\mathcal{O}}_{\lambda}, \bar{\mathbf{Q}}_l)[\dim \mathcal{O}_{\lambda} - d'_m].$$

Remarks 4.9. (i) In this case, the Springer correspondence is given by the following diagram.

$$\coprod_{0 \leq m \leq n} (S_m \times S_{n-m})^{\wedge} \simeq \coprod_{0 \leq m \leq n} \{\mathcal{O}_{\lambda} \mid \lambda \in \mathcal{P}(\mathbf{m})\} = \mathcal{X}_{\text{uni}} / \sim.$$

(ii) In the case of the enhanced variety, $Z_G(z)$ is connected for any $z \in \mathcal{X}_{\text{uni}}$. It follows that $A_G(z) = Z_G(z)/Z_G^0(z) = \{1\}$, and so the G -equivariant simple local system on the G -orbit is only the constant sheaf $\bar{\mathbf{Q}}_l$. This situation is quite similar to the case of GL_n explained in §3.

4.10. In [S1,2], a generalization of Kostka polynomials was introduced, which are functions indexed by a pair of r -partitions of n . They are a priori rational functions in $\mathbf{Z}(t)$. Here we consider such functions associated to “limit symbols” as given in [S2, §3]. In the case where $r = 2$, it was shown in [S2] that they are actually polynomials in $\mathbf{Z}[t]$, which we denote by $K_{\lambda, \mu}(t)$ for $\lambda, \mu \in \mathcal{P}_{n,2}$. (For the definition of $K_{\lambda, \mu}(t)$, see also [LS].) By [S2, Prop. 3.3], $K_{\lambda, \mu}(t) = 0$ unless $\mu \leq \lambda$, in which case it is a monic of degree $a(\mu) - a(\lambda)$. So the modified Kostka polynomial is defined by $\tilde{K}_{\lambda, \mu}(t) = t^{a(\mu)} K_{\lambda, \mu}(t^{-1})$ as in the original case. The construction of $K_{\lambda, \mu}(t)$ is purely combinatorial, but Achar and Henderson proved in [AH, Thm. 5.2] that such Kostka polynomials can be interpreted by using the geometry of the enhanced variety \mathcal{X}_{uni} , as in the case of GL_n .

Theorem 4.11 (Achar-Henderson [AH]). *For $\lambda \in \mathcal{P}_{n,2}$, consider the intersection cohomology $K = \text{IC}(\bar{\mathcal{O}}_{\lambda}, \bar{\mathbf{Q}}_l)$ on \mathcal{X}_{uni} .*

- (i) $\mathcal{H}^i K = 0$ for odd i .
- (ii) Take $\lambda, \mu \in \mathcal{P}_{n,2}$ such that $\mu \leq \lambda$. Then for $z \in \mathcal{O}_\mu \subset \overline{\mathcal{O}_\lambda}$, we have

$$(4.10.1) \quad \tilde{K}_{\lambda,\mu}(t) = t^{a(\lambda)} \sum_{i \geq 0} (\dim \mathcal{H}_z^{2i} K) t^{2i}.$$

Note that in the formula (4.10.1) each term t^i is replaced by t^{2i} , compared to the formula (3.3.1).

§5. Exotic symmetric space \mathcal{X}^{ex}

5.1. Let V be an $2n$ -dimensional vector space over \mathbf{k} , where \mathbf{k} is an algebraically closed field of $\text{ch } \mathbf{k} \neq 2$. Put $G = GL_{2n} \simeq GL(V)$. We consider the involutive automorphism $\theta : G \rightarrow G$ defined by $\theta(g) = J^{-1}({}^t g^{-1})J$ with

$$J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}.$$

Let $H = G^\theta$ be the group of θ -fixed elements in G . Then H coincides with the symplectic group Sp_{2n} . Let $\iota : G \rightarrow G, g \mapsto g^{-1}$ be the anti-automorphism on G , and define a subset of G by

$$G^{\iota\theta} = \{g \in G \mid \theta(g) = g^{-1}\}.$$

H acts on $G^{\iota\theta}$ by the conjugation action, and the map $g \rightarrow g\theta(g)^{-1}$ induces an isomorphism $G/H \simeq G^{\iota\theta}$. Under this isomorphism, the left multiplication of H on G/H corresponds to the conjugation action of H on $G^{\iota\theta}$. Thus, instead of considering the symmetric space G/H with left H -action, we may consider the closed subvariety $G^{\iota\theta}$ of G with conjugation action of H . Put $G_{\text{uni}}^{\iota\theta} = G^{\iota\theta} \cap G_{\text{uni}}$. Then $G_{\text{uni}}^{\iota\theta}$ is an H -stable subset of $G^{\iota\theta}$, which plays a role of the unipotent variety for G/H . We consider the variety $\mathcal{X} = G^{\iota\theta} \times V$ on which H acts naturally, and put $\mathcal{X}_{\text{uni}} = G_{\text{uni}}^{\iota\theta} \times V$. $\mathcal{X} = \mathcal{X}^{\text{ex}}$ is called the **exotic symmetric space**, and \mathcal{X}_{uni} is an H -stable closed subset of \mathcal{X} . The geometry of \mathcal{X} and \mathcal{X}_{uni} was studied extensively by Kato [K1,2] and [SS1-3] from different points of view. In particular, the Springer correspondence was discussed in [K2] and [SS1]. The theory of character sheaves was developed in [SS1-3]. Let \mathfrak{g} be the Lie algebra of G . θ induces a linear involutive map $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$, and we denote by $\mathfrak{g}^{-\theta}$ the -1 eigenspace of θ . Put $\mathfrak{g}_{\text{nil}}^{-\theta} = \mathfrak{g}^{-\theta} \cap \mathfrak{g}_{\text{nil}}$, on which H acts naturally. $\mathfrak{g}_{\text{nil}}^{-\theta} \times V$ is the **exotic nilpotent cone** introduced by Kato [K1], and is isomorphic to \mathcal{X}_{uni} with H -action. The following lemma was proved in [K1].

Lemma 5.2 ([K1]). *Let $\mathcal{X}_{\text{uni}}/\sim$ be the set of H -orbits in \mathcal{X}_{uni} . Then $\mathcal{X}_{\text{uni}}/\sim$ is in bijection with the set $\mathcal{P}_{n,2}$.*

5.3. $G_{\text{uni}} \times V$ is an enhanced variety discussed in §4, and the set of G -orbits in $G_{\text{uni}} \times V$ is in bijection with $\mathcal{P}_{2n,2}$. $\mathcal{X}_{\text{uni}} = G_{\text{uni}}^{\theta} \times V$ is a subset of $G_{\text{uni}} \times V$, and the action of H on \mathcal{X}_{uni} is compatible with the action of G on $G_{\text{uni}} \times V$. The connection between G -orbits in $G_{\text{uni}} \times V$ and H -orbits in \mathcal{X}_{uni} was given by Achar-Henderson [AH, Thm. 6.1] as follows; let \mathbb{O}_{ξ} be a G -orbit in $G_{\text{uni}} \times V$ with $\xi = (\xi', \xi'') \in \mathcal{P}_{2n,2}$. Then $\mathbb{O}_{\xi} \cap \mathcal{X}_{\text{uni}} \neq \emptyset$ if and only if ξ is of the form $\xi' = \lambda' \cup \lambda'', \xi'' = \lambda'' \cup \lambda'$ for some $\lambda = (\lambda', \lambda'') \in \mathcal{P}_n$, in which case $\mathbb{O}_{\xi} \cap \mathcal{X}_{\text{uni}}$ consists of a single H -orbit. We denote this H -orbit by \mathcal{O}_{λ} . Any H -orbit is obtained in this way, and this gives a parametrization of H -orbits in \mathcal{X}_{uni} . Achar-Henderson also proved in [AH, Thm. 6.3], under this parametrization, that the closure relations for $\overline{\mathcal{O}}_{\lambda}$ are given by a similar formula as Lemma 4.6, in terms of the partial order $\mu \leq \lambda$ in $\mathcal{P}_{n,2}$.

5.4. The Springer correspondence for \mathcal{X}_{uni} was first established by [K1,2] based on the Ginzburg theory on affine Hecke algebras. After that an alternate approach based on the theory of character sheaves was done by [SS1]. Here we follow the discussion in [SS1]. Let $T \subset B$ be a θ -stable pair of a maximal torus and a Borel subgroup of G . Thus $T^{\theta} \subset B^{\theta}$ is a pair of a maximal torus and a Borel subgroup of H . We denote by $\mathcal{B}^{\theta} = H/B^{\theta}$ the flag variety of H . We fix an isotropic flag $M_1 \subset M_2 \subset \cdots \subset M_n$ in V stable by B^{θ} . Define varieties

$$\begin{aligned} \widetilde{\mathcal{X}} &= \{(x, v, gB^{\theta}) \in G^{\theta} \times V \times \mathcal{B}^{\theta} \mid g^{-1}xg \in B^{\theta}, g^{-1}v \in M_n\}, \\ \widetilde{\mathcal{X}}_{\text{uni}} &= \{(x, v, gB^{\theta}) \in \widetilde{\mathcal{X}} \mid x \in G_{\text{uni}}^{\theta}\} \end{aligned}$$

and define a map $\pi : \widetilde{\mathcal{X}} \rightarrow \mathcal{X}$ by $(x, v, gB^{\theta}) \mapsto (x, v)$. Then $\pi^{-1}(\mathcal{X}_{\text{uni}}) = \widetilde{\mathcal{X}}_{\text{uni}}$, and we define $\pi_1 : \widetilde{\mathcal{X}}_{\text{uni}} \rightarrow \mathcal{X}_{\text{uni}}$ as the restriction of π on $\widetilde{\mathcal{X}}_{\text{uni}}$. π, π_1 are proper surjective maps, and $\widetilde{\mathcal{X}}, \widetilde{\mathcal{X}}_{\text{uni}}$ are smooth, irreducible varieties.

Let $W_n = S_n \ltimes (\mathbf{Z}/2\mathbf{Z})^n$ be the Weyl group of type C_n . Thus W_n is the Weyl group of H . It is known that irreducible representations of W_n are (up to isomorphism) parametrized by $\mathcal{P}_{n,2}$. We denote by \widetilde{V}_{λ} the irreducible representation of W_n corresponding to $\lambda \in \mathcal{P}_{n,2}$. The following result was proved in [SS1].

Theorem 5.5 ([SS1, Thm. 4.2]). *$\pi_* \bar{\mathbf{Q}}_l[\dim \mathcal{X}]$ is a semisimple perverse sheaf on \mathcal{X} , equipped with W_n -action, and is decomposed as*

$$(5.5.1) \quad \pi_* \bar{\mathbf{Q}}_l[\dim \mathcal{X}] \simeq \bigoplus_{\lambda \in \mathcal{P}_{n,2}} \tilde{V}_\lambda \otimes A_\lambda,$$

where A_λ is an H -equivariant simple perverse sheaf on \mathcal{X} .

Remarks 5.6. (i) In the case of the enhanced variety, for a fixed m , the local system \mathcal{L}_ρ is obtained from the finite Galois covering $\pi_m^{-1}(\mathcal{Y}_m^0) \rightarrow \mathcal{Y}_m^0$ with Galois group $S_n \times S_{n-m}$. Hence in the decomposition (4.7.1), all the simple components $\mathrm{IC}(\mathcal{X}_m, \mathcal{L}_\rho)[\dim \mathcal{X}_m]$ have the same support \mathcal{X}_m . In the case of exotic symmetric space, for each m such that $0 \leq m \leq n$, one can define a variety \mathcal{X}_m by $\mathcal{X}_m = \bigcup_{g \in H} g(B^\theta \times M_m)$. Then in the formula (5.5.1), the support of A_λ runs over all \mathcal{X}_m . Incidentally, the construction of W_n -action on $\pi_* \bar{\mathbf{Q}}_l$ is more complicated than the enhanced case (see the discussion in 7.4).

(ii) Either in the enhanced case or the exotic case, we have a filtration $\mathcal{X}_1 \subset \mathcal{X}_2 \subset \cdots \subset \mathcal{X}_n = \mathcal{X}$. The map $\pi : \mathcal{X} \rightarrow \mathcal{X}$ in the exotic case corresponds to the map π_n in the enhanced case. In the exotic case, we have only to consider the map π , but in the enhanced case, the map π_n does not reflect the general situation since the condition $g^{-1}v \in M_n = V$ is meaningless.

The following result was first proved by [K1, Thm. 8.3], [K2, Thm. G], and then reproved by [SS1, Thm. 5.4, Thm. 7.1] by a different method, which gives the Springer correspondence for the exotic case.

Theorem 5.7 ([K1, 2], [SS1]). $(\pi_1)_* \bar{\mathbf{Q}}_l[\dim \mathcal{X}_{\mathrm{uni}}]$ is a semisimple perverse sheaf on $\mathcal{X}_{\mathrm{uni}}$, equipped with W_n -action, and is decomposed as

$$(\pi_1)_* \bar{\mathbf{Q}}_l[\dim \mathcal{X}_{\mathrm{uni}}] \simeq \bigoplus_{\lambda \in \mathcal{P}_{n,2}} \tilde{V}_\lambda \otimes \mathrm{IC}(\bar{\mathcal{O}}_\lambda, \bar{\mathbf{Q}}_l)[\dim \mathcal{O}_\lambda].$$

Moreover, $A_\lambda|_{\mathcal{X}_{\mathrm{uni}}} \simeq \mathrm{IC}(\bar{\mathcal{O}}_\lambda, \bar{\mathbf{Q}}_l)$ up to shift.

5.8. We shall consider the Springer fibre for \mathcal{X} . For $z = (x, v) \in \mathcal{X}$, put

$$\mathcal{B}_z^\theta = \{gB^\theta \in \mathcal{B}^\theta \mid g^{-1}xg \in B^\theta, g^{-1}v \in M_n\}.$$

Then \mathcal{B}_z^θ is a closed subvariety of \mathcal{B}^θ isomorphic to $\pi^{-1}(z)$, and is called the Springer fibre of $z \in \mathcal{X}$. Put $d_\lambda = (\dim \mathcal{X}_{\mathrm{uni}} - \dim \mathcal{O}_\lambda)/2$. As a corollary to Theorem 5.7, we have

Corollary 5.9. (i) $\dim \mathcal{B}_z^\theta = d_\lambda$ for $z \in \mathcal{O}_\lambda$.

(ii) $H^{2d_\lambda}(\mathcal{B}_z^\theta, \bar{\mathbf{Q}}_l) \simeq \tilde{V}_\lambda$ as W_n -modules for $z \in \mathcal{O}_\lambda$. The assignment $\mathcal{O}_\lambda \mapsto H^{2d_\lambda}(\mathcal{B}_z^\theta, \bar{\mathbf{Q}}_l)$ gives a bijective correspondence

$$(5.9.1) \quad \mathcal{X}_{\mathrm{uni}} / \sim \simeq W_n^\wedge.$$

Remark 5.10. Compared to the enhanced case (see Remarks 4.9 (i)), the Springer correspondence (5.9.1) in the exotic case has a well-satisfied form. This is because we can construct representations of W_n in the exotic case, though only representations of subgroups of S_n in the enhanced case.

5.11. The relationship between the exotic symmetric space \mathcal{X}_{uni} and Kostka polynomials is studied in [K3] and [SS2]. Let $K_{\lambda, \mu}(t)$ be the Kostka polynomial associated to $\lambda, \mu \in \mathcal{P}_{n,2}$ as discussed in 4.10. We have the following result.

Theorem 5.12 ([K3, Thm. E], [SS2, Thm. 5.7]). *For $\lambda \in \mathcal{P}_{n,2}$, consider the intersection cohomology $K = \text{IC}(\overline{\mathcal{O}}_\lambda, \mathbf{Q}_l)$ on \mathcal{X}_{uni} .*

- (i) $\mathcal{H}^i K = 0$ unless $i \equiv 0 \pmod{4}$.
- (ii) Take $\lambda, \mu \in \mathcal{P}_{n,2}$ such that $\mu \leq \lambda$. Then for $z \in \mathcal{O}_\mu \subset \overline{\mathcal{O}}_\lambda$, we have

$$(5.12.1) \quad \tilde{K}_{\lambda, \mu}(t) = t^{a(\lambda)} \sum_{i \geq 0} (\dim \mathcal{H}_z^{2i} K) t^i.$$

Remarks 5.13. (i) The formula (5.12.1) was first conjectured by Achar and Henderson ([AH, Conjecture 6.4]). An idea for the proof suggested by them in [AH] was carried out by Kato (in the case where $\mathbf{k} = \mathbf{C}$). He showed in [K3, Thm. E], for each $z \in \mathcal{O}_\mu$, that the cohomology ring $H^\bullet(\mathcal{B}_z^\theta, \mathbf{C})$ has a De Concini-Procesi type interpretation as in the case of GL_n ([DP]), i.e., there exists a graded algebra isomorphism between $H^\bullet(\mathcal{B}_z^\theta, \mathbf{C})$ and $R_\mu = \mathbf{C}[x_1, \dots, x_n]/I_\mu$, compatible with the action of W_n , where I_μ is the ideal of all polynomials $p(x_1, \dots, x_n)$ such that $p(\partial/\partial x_1, \dots, \partial/\partial x_n)$ annihilates the Specht module S_μ realized in the homogeneous component of $\mathbf{C}[x_1, \dots, x_n]$ of degree $a(\mu)$. Let R_μ^i be the W_n -module obtained as the i -th homogeneous part of R_μ . It was conjectured in [S2, 3.13] that for any irreducible representation \tilde{V}_λ of W_n , we have

$$\sum_{i \geq 0} \langle R_\mu^i, \tilde{V}_\lambda \rangle_{W_n} t^i = \tilde{K}_{\lambda, \mu}(t).$$

Kato proved this conjecture in [K3, Thm. E], which provides a proof of (5.12.1), combined with the purity result ([K3, Cor. 5.3]).

The idea of the proof employed in [SS2], which was also suggested in [AH], is to construct an analogue of the theory of character sheaves on \mathcal{X} , and to use the orthogonality relations for Green functions.

(ii) The discussion on \mathcal{X}_{uni} makes sense if we restrict ourselves to the symmetric space $G_{\text{uni}}^{\iota\theta} \simeq \mathfrak{g}_{\text{nil}}^{-\theta}$ itself. In fact, the set of H -orbits in $G^{\iota\theta}$ is in bijection with the set \mathcal{P}_n , and the orbit $\mathcal{O}_{(-, \lambda'')}$ in \mathcal{X}_{uni} gives an H -orbit in

$G_{\text{uni}}^{\iota\theta}$, which we denote by $\mathcal{O}_{\lambda''}$ with $\lambda'' \in \mathcal{P}_n$. Thus $K = \text{IC}(\overline{\mathcal{O}}_{\lambda}, \bar{\mathbf{Q}}_l)$ coincides with the intersection cohomology $K^{\text{sym}} = \text{IC}(\overline{\mathcal{O}}_{\lambda''}, \bar{\mathbf{Q}}_l)$ on $G_{\text{uni}}^{\iota\theta}$. Under this setup, the following result was proved by Henderson [H, Thm. 6.3], and reproved by [SS2, Thm. 5.10]; $\mathcal{H}^i K^{\text{sym}} = 0$ unless $i \equiv 0 \pmod{4}$, and

$$(5.13.1) \quad \tilde{K}_{\lambda'', \mu''}(t^2) = t^{2n(\lambda'')} \sum_{i \geq 0} (\dim \mathcal{H}_x^{2i} K^{\text{sym}}) t^i$$

for $x \in \mathcal{O}_{\mu''}$, where $\tilde{K}_{\lambda'', \mu''}(t)$ is the original (modified) Kostka polynomial associated to $\lambda'', \mu'' \in \mathcal{P}_n$. Note that the modulo 4 vanishing of the cohomology sheaf $\mathcal{H}^i K^{\text{sym}}$ was first noticed by Grojnowski in his thesis [Gr].

Note that it is known by [AH, Cor. 5.3 (ii)] that $\tilde{K}_{\lambda, \mu}(t) = t^n \tilde{K}_{\lambda'', \mu''}(t^2)$ for such λ, μ , hence (5.13.1) is obtained as the special case of (5.12.1).

§6. Exotic symmetric space of higher level

6.1. We follow the notation in §5. For an integer $r \geq 2$, consider the varieties

$$\mathcal{X}' = G^{\iota\theta} \times V^{r-1} \supset \mathcal{X}'_{\text{uni}} = G_{\text{uni}}^{\iota\theta} \times V^{r-1}$$

with diagonal action of H on \mathcal{X}' , $\mathcal{X}'_{\text{uni}}$. \mathcal{X}' , $\mathcal{X}'_{\text{uni}}$ are a natural generalization of the exotic symmetric space studied in §5. But it occurs a crucial difference when we consider the general r , i.e.,

(6.1.1) If $r \geq 3$, $\mathcal{X}'_{\text{uni}}$ has infinitely many H -orbits.

In fact, since $\dim G_{\text{uni}}^{\iota\theta} = 2n^2 - 2n$, we have

$$\dim \mathcal{X}'_{\text{uni}} = 2n^2 - 2n + (r-1)2n > \dim H = 2n^2 + n$$

if $r \geq 3$.

Let us define varieties

$$\begin{aligned} \widetilde{\mathcal{X}} &= \{(x, \mathbf{v}, gB^\theta) \in G^{\iota\theta} \times V^{r-1} \times \mathcal{B}^\theta \mid g^{-1}xg \in B^{\iota\theta}, g^{-1}\mathbf{v} \in M_n^{r-1}\}, \\ \mathcal{X} &= \bigcup_{g \in H} g(B^{\iota\theta} \times M_n^{r-1}), \\ \widetilde{\mathcal{X}}_{\text{uni}} &= \{(x, \mathbf{v}, gB^\theta) \in \widetilde{\mathcal{X}} \mid x \in G_{\text{uni}}^{\iota\theta}\}, \\ \mathcal{X}_{\text{uni}} &= \{(x, \mathbf{v}) \in \mathcal{X} \mid x \in G_{\text{uni}}^{\iota\theta}\} \end{aligned}$$

and a map $\pi : \widetilde{\mathcal{X}} \rightarrow \mathcal{X}$ by $(x, \mathbf{v}, gB^\theta) \mapsto (x, \mathbf{v})$. We also define $\pi_1 : \widetilde{\mathcal{X}}_{\text{uni}} \rightarrow \mathcal{X}_{\text{uni}}$ by the restriction of π on $\widetilde{\mathcal{G}}_{\text{uni}}$. The maps π, π_1 are proper surjective.

Hence \mathcal{X} is a closed subset of $\mathcal{X}' = G^{t\theta} \times V^{r-1}$, and similarly for \mathcal{X}_{uni} . \mathcal{X} is called the **exotic symmetric space of level r** . Note that the map $\pi : \mathcal{X} \rightarrow \mathcal{X}'$ is not necessarily surjective if $r \geq 3$. So we replace \mathcal{X}' by the image $\pi(\widetilde{\mathcal{X}}) = \mathcal{X}$, and $\mathcal{X}'_{\text{uni}}$ by $\pi_1(\widetilde{\mathcal{X}}_{\text{uni}})$. But \mathcal{X}_{uni} has still infinitely many H -orbits. The Springer correspondence for \mathcal{X}_{uni} was established in [S3], which will be discussed in next section. In this section, we prepare some notations.

6.2. We consider certain subvarieties of $\mathcal{X}, \mathcal{X}_{\text{uni}}$, as in the case of the enhanced variety. Put

$$\begin{aligned}\mathcal{Q}_{n,r} &= \{\mathbf{m} = (m_1, \dots, m_r) \in \mathbf{Z}_{\geq 0}^r \mid \sum m_i = n\}, \\ \mathcal{Q}_{n,r}^0 &= \{\mathbf{m} \in \mathcal{Q}_{n,r} \mid m_r = 0\}.\end{aligned}$$

By fixing $\mathbf{m} \in \mathcal{Q}_{n,r}$, define p_1, \dots, p_r by $p_k = m_1 + \dots + m_k$.
For each $\mathbf{m} \in \mathcal{Q}_{n,r}$, put

$$\begin{aligned}\widetilde{\mathcal{X}}_{\mathbf{m}} &= \{(x, \mathbf{v}, gB^\theta) \in G^{t\theta} \times V^{r-1} \times \mathcal{B}^\theta \mid g^{-1}xg \in B^{t\theta}, g^{-1}\mathbf{v} \in \prod_{i=1}^{r-1} M_{p_i}\} \\ \mathcal{X}_{\mathbf{m}} &= \bigcup_{g \in H} g(B^{t\theta} \times \prod_{i=1}^{r-1} M_{p_i}) \\ \pi^{(\mathbf{m})} : \widetilde{\mathcal{X}}_{\mathbf{m}} &\rightarrow \mathcal{X}_{\mathbf{m}} \quad (x, \mathbf{v}, gB^\theta) \mapsto (x, \mathbf{v})\end{aligned}$$

$\widetilde{\mathcal{X}}_{\mathbf{m}}$ is smooth, irreducible, and $\pi^{(\mathbf{m})}$ is proper, surjective. Hence $\mathcal{X}_{\mathbf{m}}$ is a closed subset of $G^{t\theta} \times V^{r-1}$. Note that if $\mathbf{m} = (n, 0, \dots, 0)$, then $\widetilde{\mathcal{X}}_{\mathbf{m}} = \mathcal{X}$ and $\mathcal{X}_{\mathbf{m}} = \mathcal{X}$. The dimension of the varieties $\widetilde{\mathcal{X}}_{\mathbf{m}}, \mathcal{X}_{\mathbf{m}}$ are computed as follows;

Lemma 6.3. (i) $\dim \widetilde{\mathcal{X}}_{\mathbf{m}} = 2n^2 + \sum_{i=1}^r (r-i)m_i$.
(ii) $\dim \mathcal{X}_{\mathbf{m}} = 2n^2 + \sum_{i=1}^r (r-i)m_i - m_r$. In particular, if $m_r = 0$, i.e., if $\mathbf{m} \in \mathcal{Q}_{n,r}^0$, then $\dim \widetilde{\mathcal{X}}_{\mathbf{m}} = \dim \mathcal{X}_{\mathbf{m}}$.

The condition $\dim \widetilde{\mathcal{X}}_{\mathbf{m}} = \dim \mathcal{X}_{\mathbf{m}}$ plays an important role in later discussions (the semi-smallness of the map $\pi^{(\mathbf{m})}$). So in order to guarantee this condition, we pose the assumption $\mathbf{m} \in \mathcal{Q}_{n,r}^0$ in some situations.

6.4. Let $W_{n,r} = S_n \ltimes (\mathbf{Z}/r\mathbf{Z})^n$ be the complex reflection group $G(r, 1, n)$. It is well-known that the set $W_{n,r}^\wedge$ of irreducible representations of $W_{n,r}$ is parametrized by the set $\mathcal{P}_{n,r}$. We denote by \widetilde{V}_λ the irreducible representation of $W_{n,r}$ corresponding to $\lambda \in \mathcal{P}_{n,r}$. For later use, we review the construction of \widetilde{V}_λ . Let $\zeta \in \bar{\mathbf{Q}}_l^*$ be a primitive r -th root of unity. Define a linear character

$\tau_i : \mathbf{Z}/r\mathbf{Z} \rightarrow \bar{\mathbf{Q}}_l^*$ by $1 + r\mathbf{Z} \mapsto \zeta^{i-1}$ for $1 \leq i \leq r$. For $\mathbf{m} \in \mathcal{Q}_{n,r}$, put $S_{\mathbf{m}} = S_{m_1} \times S_{m_2} \times \cdots \times S_{m_r} \subset S_n$. For each i , put $\tilde{S}_{m_i} = S_{m_i} \ltimes (\mathbf{Z}/r\mathbf{Z})^{m_i}$, and consider a subgroup $\tilde{S}_{\mathbf{m}} = \tilde{S}_{m_1} \times \cdots \times \tilde{S}_{m_r}$ of $W_{n,r}$.

An irreducible representation ρ of $S_{\mathbf{m}}$ can be written as $\rho = \rho_1 \boxtimes \cdots \boxtimes \rho_r$ with $\rho_i \in S_{m_i}^\wedge$. We extend ρ_i to the irreducible representation $\tilde{\rho}_i$ on \tilde{S}_{m_i} by defining the action of $(\mathbf{Z}/r\mathbf{Z})^{m_i}$ on the space ρ_i by $\tau_i^{\otimes m_i}$. Let $\tilde{\rho} = \tilde{\rho}_1 \boxtimes \cdots \boxtimes \tilde{\rho}_r \in \tilde{S}_{\mathbf{m}}^\wedge$, and put

$$\tilde{V}_\rho = \text{Ind}_{\tilde{S}_{\mathbf{m}}}^{W_{n,r}} \tilde{\rho}.$$

Then \tilde{V}_ρ gives an irreducible representation of $W_{n,r}$. Since $S_{m_i}^\wedge \simeq \mathcal{P}_{m_i}$, we can write $\rho_i = V_{\lambda^{(i)}}$ for $\lambda^{(i)} \in \mathcal{P}_{m_i}$. Then $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)}) \in \mathcal{P}_{n,r}$, and we write $\rho = \rho_\lambda$ and $\tilde{V}_\rho = \tilde{V}_\lambda$. This gives the required parametrisation $W_{n,r}^\wedge \simeq \mathcal{P}_{n,r}$.

6.5. Here we introduce a partition of $W_{n,r}^\wedge$ corresponding to $\mathcal{Q}_{n,r}^0$. Assume that $\mathbf{m} \in \mathcal{Q}_{n,r}^0$, namely $\mathbf{m} = (m_1, \dots, m_{r-1}, 0)$. For $0 \leq k \leq m_{r-1}$, put $\mathbf{m}(k) = (m_1, \dots, m_{r-2}, k, k') \in \mathcal{Q}_{n,r}$ with $k + k' = m_{r-1}$. Put $S_{\mathbf{m}(k)} = S_{m_1} \times \cdots \times S_{m_{r-2}} \times S_k \times S_{k'}$, and define

$$(W_{n,r}^\wedge)_{\mathbf{m}} = \{\tilde{V}_\rho \mid \rho \in S_{\mathbf{m}(k)}^\wedge \text{ for } 0 \leq k \leq m_{r-1}\} \subset W_{n,r}^\wedge.$$

Then we have a partition of $W_{n,r}^\wedge$

$$W_{n,r}^\wedge = \coprod_{\mathbf{m} \in \mathcal{Q}_{n,r}^0} (W_{n,r}^\wedge)_{\mathbf{m}}.$$

For $\mathbf{m} \in \mathcal{Q}_{n,r}$, let $\mathcal{P}(\mathbf{m})$ be the subset of $\mathcal{P}_{n,r}$ consisting of λ such that $|\lambda^{(i)}| = m_i$ for $i = 1, \dots, r$. For $\mathbf{m} \in \mathcal{Q}_{n,r}^0$, put

$$\widetilde{\mathcal{P}}(\mathbf{m}) = \coprod_{0 \leq k \leq m_{r-1}} \mathcal{P}(\mathbf{m}(k)).$$

Thus $\widetilde{\mathcal{P}}(\mathbf{m})$ is the subset of $\mathcal{P}_{n,r}$ consisting of λ such that $|\lambda^{(i)}| = i$ for $i = 1, \dots, r-2$. It is easy to see that

$$(6.5.1) \quad (W_{n,r}^\wedge)_{\mathbf{m}} = \{\tilde{V}_\lambda \mid \lambda \in \widetilde{\mathcal{P}}(\mathbf{m})\}.$$

6.6. Recall that W_n ($= W_{n,2}$ in the notation of 6.4) is the Weyl group of type C_n . For $\mathbf{m} \in \mathcal{Q}_{n,r}^0$, we define a parabolic subgroup $W_{\mathbf{m}}^{\natural}$ of W_n by

$$W_{\mathbf{m}}^{\natural} = S_{m_1} \times \cdots \times S_{m_{r-2}} \times W_{m_{r-1}} \subset W_n.$$

For $\rho = \rho_1 \boxtimes \cdots \boxtimes \rho_r \in S_{\mathbf{m}(k)}^\wedge$ we define an irreducible $W_{\mathbf{m}}^\natural$ -module V_ρ^\natural by

$$V_\rho^\natural = \rho_1 \boxtimes \cdots \boxtimes \rho_{r-2} \boxtimes \psi_{r-1},$$

where

$$\psi_{r-1} = \text{Ind}_{\tilde{S}_k \times \tilde{S}_{k'}}^{W_{m_{r-1}}}(\tilde{\rho}_{r-1} \boxtimes \tilde{\rho}_r)$$

(apply the construction of \tilde{V}_λ for the case $r = 2$). Then we have a natural bijection

$$(W_{\mathbf{m}}^\natural)^\wedge \simeq \coprod_{0 \leq k \leq m_{r-1}} S_{\mathbf{m}(k)}^\wedge \simeq (W_{n,r}^\wedge)_{\mathbf{m}}$$

through $V_\rho^\natural \leftrightarrow \rho \leftrightarrow \tilde{V}_\rho$. We denote by V_λ^\natural the irreducible representation of $W_{n,r}^\natural$ corresponding to \tilde{V}_λ . It is easy to see that

$$(6.6.1) \quad (W_{\mathbf{m}}^\natural)^\wedge = \{V_\lambda^\natural \mid \lambda \in \tilde{\mathcal{P}}(\mathbf{m})\}.$$

§7. Springer correspondence for \mathcal{X}^{ex} of higher level

7.1 In this section, we shall discuss about the Springer correspondence for the exotic symmetric space \mathcal{X} of level r based on [S3]. First we generalize Theorem 5.5 to the case where r is arbitrary. Take $\mathbf{m} \in \mathcal{Q}_{n,r}^0$. Recall the map $\pi^{(\mathbf{m})} : \tilde{\mathcal{X}}_{\mathbf{m}} \rightarrow \mathcal{X}_{\mathbf{m}}$. We define a map $\pi_{\mathbf{m}} : \pi^{-1}(\mathcal{X}_{\mathbf{m}}) \rightarrow \mathcal{X}_{\mathbf{m}}$ as the restriction of π on $\pi^{-1}(\mathcal{X}_{\mathbf{m}})$. Note that $\tilde{\mathcal{X}}_{\mathbf{m}} \subset \pi^{-1}(\mathcal{X}_{\mathbf{m}})$ since

$$\begin{aligned} \pi^{-1}(\mathcal{X}_{\mathbf{m}}) &= \{(x, \mathbf{v}, gB^\theta) \mid (x, \mathbf{v}) \in \mathcal{X}_{\mathbf{m}}, g^{-1}xg \in B^{i\theta}, g^{-1}\mathbf{v} \in M_n^{r-1}\}, \\ \tilde{\mathcal{X}}_{\mathbf{m}} &= \{(x, \mathbf{v}, gB^\theta) \in \pi^{-1}(\mathcal{X}_{\mathbf{m}}) \mid g^{-1}\mathbf{v} \in \prod_{i=1}^{r-1} M_{p_i}\}. \end{aligned}$$

Put $d_{\mathbf{m}} = \dim \mathcal{X}_{\mathbf{m}}$. The following two results are both generalizations of Theorem 5.5 in the case $r = 2$. (Note that $\pi_{\mathbf{m}}$ in this paper is written as $\bar{\pi}_{\mathbf{m}}$ in [S3]).

Theorem 7.2 ([S3, Thm. 3.2]). *Assume that $\mathbf{m} \in \mathcal{Q}_{n,r}^0$. Then $\pi_*^{(\mathbf{m})} \bar{\mathbf{Q}}_l[d_{\mathbf{m}}]$ is a semisimple perverse sheaf on $\mathcal{X}_{\mathbf{m}}$, equipped with $W_{\mathbf{m}}^\natural$ -action, and is decomposed as*

$$\pi_*^{(\mathbf{m})} \bar{\mathbf{Q}}_l[d_{\mathbf{m}}] \simeq \bigoplus_{0 \leq k \leq m_{r-1}} \bigoplus_{\rho \in S_{\mathbf{m}(k)}^\wedge} V_\rho^\natural \otimes \text{IC}(\mathcal{X}_{\mathbf{m}(k)}, \mathcal{L}_\rho)[d_{\mathbf{m}(k)}],$$

where \mathcal{L}_ρ is a simple local system on a certain open dense subset of $\mathcal{X}_{\mathbf{m}(k)}$ associated to $\rho \in S_{\mathbf{m}(k)}^\wedge$.

Theorem 7.3 ([S3, Thm. 2.2]). *Assume that $\mathbf{m} \in \mathcal{Q}_{n,r}^0$. Then $(\pi_{\mathbf{m}})_* \bar{\mathbf{Q}}_l[d_{\mathbf{m}}]$ is a semisimple perverse sheaf on $\mathcal{X}_{\mathbf{m}}$, equipped with $W_{n,r}$ -action, and is decomposed as*

$$(\pi_{\mathbf{m}})_* \bar{\mathbf{Q}}_l[d_{\mathbf{m}}] \simeq \bigoplus_{0 \leq k \leq m_{r-1}} \bigoplus_{\rho \in S_{\mathbf{m}(k)}^\wedge} \tilde{V}_\rho \otimes \mathrm{IC}(\mathcal{X}_{\mathbf{m}(k)}, \mathcal{L}_\rho)[d_{\mathbf{m}(k)}].$$

7.4. The construction of $W_{n,r}$ -action on $(\pi_{\mathbf{m}})_* \bar{\mathbf{Q}}_l$ is a natural generalization of the construction of W_n -action in the case where $r = 2$. Here we give some explanation on the construction of $W_{n,r}$ -action and on the definition of local systems \mathcal{L}_ρ involved in the theorems.

Let $T_{\mathrm{reg}}^{\iota\theta}$ be the set of regular semisimple elements in $T^{\iota\theta}$, i.e., the set of elements such that all the eigenvalues have multiplicity 2. Let

$$G_{\mathrm{reg}}^{\iota\theta} = \bigcup_{g \in H} g T_{\mathrm{reg}}^{\iota\theta} g^{-1} \subset G^{\iota\theta}$$

be the set of regular semisimple elements in $G^{\iota\theta}$, which is an open dense subset of $G^{\iota\theta}$. For $\mathbf{m} \in \mathcal{Q}_{n,r}$, we define

$$\begin{aligned} \widetilde{\mathcal{Y}}_{\mathbf{m}} &= \{(x, \mathbf{v}, gB^\theta) \in \widetilde{\mathcal{X}}_{\mathbf{m}} \mid x \in G_{\mathrm{reg}}^{\iota\theta}\}, \\ \mathcal{Y}_{\mathbf{m}} &= \{(x, \mathbf{v}) \in \mathcal{X}_{\mathbf{m}} \mid x \in G_{\mathrm{reg}}^{\iota\theta}\}, \\ \psi^{(\mathbf{m})} : \widetilde{\mathcal{Y}}_{\mathbf{m}} &\rightarrow \mathcal{Y}_{\mathbf{m}} \quad (x, \mathbf{v}, gB^\theta) \mapsto (x, \mathbf{v}). \end{aligned}$$

Then $\mathcal{Y}_{\mathbf{m}}$ is an open dense subset in $\mathcal{X}_{\mathbf{m}}$, and $\psi^{(\mathbf{m})}$ is the restriction of $\pi^{(\mathbf{m})}$ on $\widetilde{\mathcal{Y}}_{\mathbf{m}}$. We put $\psi : \widetilde{\mathcal{Y}} \rightarrow \mathcal{Y}$, where $\widetilde{\mathcal{Y}} = \widetilde{\mathcal{Y}}_{\mathbf{m}}$, $\mathcal{Y} = \mathcal{Y}_{\mathbf{m}}$ for $\mathbf{m} = (n, 0, \dots, 0)$. As in the case of $\pi_{\mathbf{m}}$, we define $\psi_{\mathbf{m}} : \psi^{-1}(\mathcal{Y}_{\mathbf{m}}) \rightarrow \widetilde{\mathcal{Y}}_{\mathbf{m}}$ as the restriction of ψ on $\psi^{-1}(\mathcal{Y}_{\mathbf{m}})$. Hence $\widetilde{\mathcal{Y}}_{\mathbf{m}} \subset \psi^{-1}(\mathcal{Y}_{\mathbf{m}})$. Note that $\widetilde{\mathcal{Y}}_{\mathbf{m}}$ is expressed as

$$\widetilde{\mathcal{Y}}_{\mathbf{m}} \simeq H \times^{B^\theta \cap Z_H(T^{\iota\theta})} (T_{\mathrm{reg}}^{\iota\theta} \times \prod_i M_{p_i}).$$

Define a partial order on $\mathcal{Q}_{n,r}$ by $\mathbf{m}' \leq \mathbf{m}$ if and only if $p'_i \leq p_i$ ($1 \leq i \leq r$), where $p_i = m_1 + \dots + m_i$, $p'_i = m'_1 + \dots + m'_i$. Define $\mathcal{Y}_{\mathbf{m}}^0 = \mathcal{Y}_{\mathbf{m}} \setminus \bigcup_{\mathbf{m}' < \mathbf{m}} \mathcal{Y}_{\mathbf{m}'}$. Then $\mathcal{Y}_{\mathbf{m}}^0$ is an open dense subset of $\mathcal{Y}_{\mathbf{m}}$. We define a subvariety $\widetilde{\mathcal{Y}}_{\mathbf{m}}^0$ of $\widetilde{\mathcal{Y}}_{\mathbf{m}}$ by

$$\widetilde{\mathcal{Y}}_{\mathbf{m}}^0 = H \times^{B^\theta \cap Z_H(T^{\iota\theta})} (T_{\text{reg}}^{\iota\theta} \times \prod_i M_{p_i}^0),$$

where $M_{p_i}^0$ is defined as in 4.7. S_n acts naturally on $\psi^{-1}(\mathcal{Y}_{\mathbf{m}}^0)$, and we have

$$(7.4.1) \quad \psi^{-1}(\mathcal{Y}_{\mathbf{m}}^0) \simeq \coprod_{w \in S_n/S_{\mathbf{m}}} w(\widetilde{\mathcal{Y}}_{\mathbf{m}}^0),$$

which gives the decomposition of $\psi^{-1}(\mathcal{Y}_{\mathbf{m}}^0)$ into irreducible components.

We define a variety

$$\widehat{\mathcal{Y}}_{\mathbf{m}}^0 = H \times^{Z_H(T^{\iota\theta})_{\mathbf{m}}} (T_{\text{reg}}^{\iota\theta} \times \prod_i M_{p_i}^0),$$

where

$$\begin{aligned} Z_H(T^{\iota\theta}) &\simeq SL_2 \times \cdots \times SL_2 \quad (n\text{-times}), \\ Z_H(T^{\iota\theta}) \cap B^\theta &\simeq B_2 \times \cdots \times B_2 \quad (B_2: \text{Borel of } SL_2), \\ Z_H(T^{\iota\theta})_{\mathbf{m}} &\simeq B_2 \times \cdots \times B_2 \times \underbrace{SL_2 \times \cdots \times SL_2}_{\text{last } m_r\text{-factors}}. \end{aligned}$$

Then the map $\psi^{(\mathbf{m})} : \widetilde{\mathcal{Y}}_{\mathbf{m}}^0 \rightarrow \mathcal{Y}_{\mathbf{m}}^0$ is decomposed as

$$\psi^{(\mathbf{m})} : \widetilde{\mathcal{Y}}_{\mathbf{m}}^0 \xrightarrow{\xi_{\mathbf{m}}} \widehat{\mathcal{Y}}_{\mathbf{m}}^0 \xrightarrow{\eta_{\mathbf{m}}} \mathcal{Y}_{\mathbf{m}}^0,$$

where $\xi_{\mathbf{m}}$ is a locally trivial fibration with fibre $\simeq \mathbf{P}_1^{m_r}$, and $\eta_{\mathbf{m}}$ is a finite Galois covering with group $S_{\mathbf{m}}$. It follows that

$$(7.4.2) \quad (\xi_{\mathbf{m}})_* \bar{\mathbf{Q}}_l \simeq H^\bullet(\mathbf{P}_1^{m_r}) \otimes \bar{\mathbf{Q}}_l,$$

$$(7.4.3) \quad (\eta_{\mathbf{m}})_* \bar{\mathbf{Q}}_l \simeq \bigoplus_{\rho \in S_{\mathbf{m}}^\wedge} \rho \otimes \mathcal{L}_\rho,$$

where \mathcal{L}_ρ is a simple local sytem on $\mathcal{Y}_{\mathbf{m}}^0$. We define $\psi_{\mathbf{m}}^0 : \psi^{-1}(\mathcal{Y}_{\mathbf{m}}^0) \rightarrow \mathcal{Y}_{\mathbf{m}}^0$ by the restriciton of $\psi : \widetilde{\mathcal{Y}} \rightarrow \mathcal{Y}$. In view of (7.4.1) \sim (7.4.3), we have

$$(\psi_{\mathbf{m}}^0)_* \bar{\mathbf{Q}}_l \simeq \bigoplus_{\rho \in S_{\mathbf{m}}^\wedge} \text{Ind}_{S_{\mathbf{m}}}^{S_n} (H^\bullet(\mathbf{P}_1^{m_r}) \otimes \rho) \otimes \mathcal{L}_\rho$$

$$\simeq \bigoplus_{\rho \in S_{\mathbf{m}}^{\wedge}} \text{Ind}_{\tilde{S}_{\mathbf{m}}}^{W_{n,r}} (H^{\bullet}(\mathbf{P}_1^{m_r}) \otimes \tilde{\rho}') \otimes \mathcal{L}_{\rho}.$$

In the second formula, one can define an action of $(\mathbf{Z}/r\mathbf{Z})^n$ on $H^{\bullet}(\mathbf{P}_1^{m_r})$ and on ρ so that $H^{\text{top}}(\mathbf{P}_1^{m_r}) \otimes \tilde{\rho}' \simeq \tilde{\rho} \in \tilde{S}_{\mathbf{m}}^{\wedge}$. Thus $(\psi_{\mathbf{m}}^0)_* \bar{\mathbf{Q}}_l$ turns out to be a local system equipped with $W_{n,r}$ -action. By using the decomposition $\mathcal{Y}_{\mathbf{m}} = \coprod_{\mathbf{m}' \leq \mathbf{m}} \mathcal{Y}_{\mathbf{m}'}^0$, we can determine the decomposition of $(\psi_{\mathbf{m}})_* \bar{\mathbf{Q}}_l[d_{\mathbf{m}}]$ as follows;

$$(7.4.4) \quad (\psi_{\mathbf{m}})_* \bar{\mathbf{Q}}_l[d_{\mathbf{m}}] \simeq \bigoplus_{0 \leq k \leq m_{r-1}} \bigoplus_{\rho \in S_{\mathbf{m}(k)}^{\wedge}} \tilde{V}_{\rho} \otimes \text{IC}(\mathcal{Y}_{\mathbf{m}(k)}, \mathcal{L}_{\rho})[d_{\mathbf{m}(k)}].$$

This formula corresponds to the formula (2.1.1) in the original case, and one can prove that $(\pi_{\mathbf{m}})_* \bar{\mathbf{Q}}_l[d_{\mathbf{m}}]$ coincides with the intermediate extension of $(\psi_{\mathbf{m}})_* \bar{\mathbf{Q}}_l[d_{\mathbf{m}}]$ (note that $\mathcal{Y}_{\mathbf{m}(k)}$ is open dense in $\mathcal{X}_{\mathbf{m}(k)}$).

7.5. We now discuss about the unipotent variety \mathcal{X}_{uni} . Recall the map $\pi_1 : \tilde{\mathcal{X}}_{\text{uni}} \rightarrow \mathcal{X}_{\text{uni}}$. For $\mathbf{m} \in \mathcal{Q}_{n,r}$, put

$$\begin{aligned} \tilde{\mathcal{X}}_{\mathbf{m},\text{uni}} &= \{(x, \mathbf{v}, gB^{\theta}) \in \tilde{\mathcal{X}}_{\mathbf{m}} \mid x \in G_{\text{uni}}^{u\theta}\} \subset \tilde{\mathcal{X}}_{\text{uni}}, \\ \mathcal{X}_{\mathbf{m},\text{uni}} &= \mathcal{X}_{\mathbf{m}} \cap \mathcal{X}_{\text{uni}}, \\ \pi_1^{(\mathbf{m})} : \tilde{\mathcal{X}}_{\mathbf{m},\text{uni}} &\rightarrow \mathcal{X}_{\mathbf{m},\text{uni}}, \quad (x, \mathbf{v}, gB^{\theta}) \mapsto (x, \mathbf{v}). \end{aligned}$$

$\tilde{\mathcal{X}}_{\mathbf{m},\text{uni}}$ is smooth, irreducible, and the map $\pi_1^{(\mathbf{m})}$ is proper, surjective. We have the following result.

Proposition 7.6. (i) For each $\mathbf{m} \in \mathcal{Q}_{n,r}$,

$$\dim \tilde{\mathcal{X}}_{\mathbf{m},\text{uni}} = 2n^2 - n + \sum_{i=1}^{r-1} (r-i)m_i.$$

(ii) Assume that $\mathbf{m} \in \mathcal{Q}_{n,r}^0$. Then $\dim \tilde{\mathcal{X}}_{\mathbf{m},\text{uni}} = \dim \mathcal{X}_{\mathbf{m},\text{uni}}$.

Actually, it can be proved that if $\mathbf{m} \in \mathcal{Q}_{n,r}^0$, then $\pi_1^{(\mathbf{m})}$ is semi-small.

7.7. As remarked in (6.1.1), \mathcal{X}_{uni} has infinitely many H -orbits if $r \geq 3$. Thus we need to construct a certain set of subvarieties of \mathcal{X}_{uni} which has a similar role as the set of H -orbits in the case where $r = 2$. Since we want to establish the Springer correspondence with $W_{n,r}^{\wedge}$, these varieties must be

parametrized by $\mathcal{P}_{n,r}$. In [S3], such a variety X_λ for each $\lambda \in \mathcal{P}_{n,r}$ was constructed.

Proposition 7.8. *X_λ is a locally closed, smooth, irreducible, H -stable subvariety of \mathcal{X}_{uni} , satisfying the following properties.*

(i) *We have*

$$\dim X_\lambda = 2n^2 - 2n - 2n(\lambda) - 2n(\lambda^{(r-1)} + \lambda^{(r)}) + \sum_{i=1}^{r-1} (r-i+1)|\lambda^{(i)}|.$$

(ii) *Assume that $\mathbf{m} \in \mathcal{Q}_{n,r}^0$. Then $\overline{X}_{\lambda(\mathbf{m})} = \mathcal{X}_{\mathbf{m},\text{uni}}$, where $\lambda(\mathbf{m}) = ((m_1), (m_2), \dots, (m_{r-1}), (m_r))$.*

(iii) *Assume that $\mathbf{m} \in \mathcal{Q}_{n,r}^0$ and that $\mu \in \widetilde{\mathcal{P}}(\mathbf{m})$. Then $X_\mu \subset \mathcal{X}_{\mathbf{m},\text{uni}}$.*

(iv) *If $r = 2$, X_λ coincides with the H -orbit \mathcal{O}_λ .*

7.9. We explain the construction of X_λ . Let $\lambda \in \mathcal{P}(\mathbf{m})$ for $\mathbf{m} \in \mathcal{Q}_{n,r}$. Let P be the θ -stable parabolic subgroup of G such that P^θ is the stabilizer of the (partial) isotropic flag $(M_{p_i})_{1 \leq i \leq r-2}$, and L the θ -stable Levi subgroup of P such that $L \supset T$. We shall define a set

$$\mathcal{M}_\lambda \subset P_{\text{uni}}^{\iota\theta} \times \left(\prod_{i=1}^{r-2} M_{p_i} \times M_{p_{r-2}}^\perp \right)$$

as follows; put $\overline{M}_{p_i} = M_{p_i}/M_{p_{i-1}}$ for $i = 1, \dots, r-2$, and $\overline{M}_{p_{r-1}} = M_{p_{r-2}}^\perp/M_{p_{r-2}}$. Take $\mathbf{v} = (v_1, \dots, v_{r-1}) \in V^{r-1}$ such that $v_i \in M_{p_i}$ for $i = 1, \dots, r-2$, and $v_{r-1} \in M_{p_{r-2}}^\perp$. Let \overline{v}_i be the image of v_i on \overline{M}_{p_i} for $i = 1, \dots, r-1$. Since $\overline{M}_{p_{r-1}}$ has a natural symplectic structure, one can define an exotic symmetric space $\mathcal{X}_{\text{uni}}^{(r-1)} = GL(\overline{M}_{p_{r-1}})_{\text{uni}}^{\iota\theta} \times \overline{M}_{p_{r-1}}$. For $i = 1, \dots, r-2$, one can define an enhanced variety $\mathcal{X}_{\text{uni}}^{(i)} = GL(\overline{M}_{p_i})_{\text{uni}} \times \overline{M}_{p_i}$, whose $GL(\overline{M}_{p_i})$ -orbits are parametrized by $\mathcal{P}_{m_i,2}$. Let $\pi_P : P_{\text{uni}}^{\iota\theta} \times V \rightarrow L_{\text{uni}}^{\iota\theta} \times V/M_{p_{r-2}}$ be the natural map. Let \mathcal{O} be the H -orbit of $z = (x, v_{r-1})$ and \mathcal{O}' be the L^θ -orbit of $z' = \pi_P(z)$. We define \mathcal{M}_λ as the set of (x, \mathbf{v}) satisfying the following conditions;

- (i) $(x|_{\overline{M}_{p_i}}, \overline{v}_i) \in \mathcal{X}_{\text{uni}}^{(i)}$ has type $(\lambda^{(i)}, \emptyset)$ for $i = 1, \dots, r-2$.
- (ii) $(x|_{\overline{M}_{p_{r-1}}}, \overline{v}_{r-1}) \in \mathcal{X}_{\text{uni}}^{(r-1)}$ has type $(\lambda^{(r-1)}, \lambda^{(r)})$.
- (iii) $\mathcal{O} \cap \pi_P^{-1}(\mathcal{O}')$ is open dense in $\pi_P^{-1}(\mathcal{O}')$.

Finally we define

$$X_\lambda = \bigcup_{g \in H} g \mathcal{M}_\lambda.$$

Remark 7.10. $\bigcup_{\lambda \in \mathcal{P}_{n,r}} X_\lambda$ covers an open dense subset of \mathcal{X}_{uni} , but it does not coincide with \mathcal{X}_{uni} if $r \geq 3$. Furthermore, X_λ 's are not mutually disjoint in general.

7.11. Recall the map $\pi_1^{(\mathbf{m})} : \widetilde{\mathcal{X}}_{\mathbf{m},\text{uni}} \rightarrow \mathcal{X}_{\mathbf{m},\text{uni}}$. We define a map $\pi_{\mathbf{m},1} : \pi_1^{-1}(\mathcal{X}_{\mathbf{m},\text{uni}}) \rightarrow \mathcal{X}_{\mathbf{m},\text{uni}}$ as the restriction of π_1 . As before, $\widetilde{\mathcal{X}}_{\mathbf{m},\text{uni}} \subset \pi_1^{-1}(\mathcal{X}_{\mathbf{m},\text{uni}})$. Since

$$(\pi_1^{(\mathbf{m})})_* \bar{\mathbf{Q}}_l \simeq (\pi^{(\mathbf{m})})_* \bar{\mathbf{Q}}_l|_{\mathcal{X}_{\mathbf{m},\text{uni}}}, \quad (\pi_{\mathbf{m},1})_* \bar{\mathbf{Q}}_l \simeq (\pi_{\mathbf{m}})_* \bar{\mathbf{Q}}_l|_{\mathcal{X}_{\mathbf{m},\text{uni}}},$$

$(\pi_1^{(\mathbf{m})})_* \bar{\mathbf{Q}}_l$ has a natural $W_{\mathbf{m}}^{\natural}$ -action inherited from $(\pi^{(\mathbf{m})})_* \bar{\mathbf{Q}}_l$, and $(\pi_{\mathbf{m},1})_* \bar{\mathbf{Q}}_l$ has a natural $W_{n,r}$ -action inherited from $(\pi_{\mathbf{m}})_* \bar{\mathbf{Q}}_l$. Put $d'_{\mathbf{m}} = \dim \mathcal{X}_{\mathbf{m},\text{uni}}$. The following results give the Springer correspondence for \mathcal{X}_{uni} , the former one is with respect to $W_{\mathbf{m}}^{\natural}$, the latter one is with respect to $W_{n,r}$.

Theorem 7.12 ([S3, Thm. 7.12, Thm. 8.17]). *Assume that $\mathbf{m} \in \mathcal{Q}_{n,r}^0$.*

- (i) $(\pi_1^{(\mathbf{m})})_* \bar{\mathbf{Q}}_l[d'_{\mathbf{m}}]$ is a semisimple perverse sheaf on $\mathcal{X}_{\mathbf{m},\text{uni}}$ equipped with $W_{\mathbf{m}}^{\natural}$ -action, and is decomposed as

$$(\pi_1^{(\mathbf{m})})_* \bar{\mathbf{Q}}_l[d'_{\mathbf{m}}] \simeq \bigoplus_{0 \leq k \leq m_{r-1}} \bigoplus_{\lambda \in \mathcal{P}(\mathbf{m}(k))} V_\lambda^{\natural} \otimes \text{IC}(\overline{X}_\lambda, \bar{\mathbf{Q}}_l)[\dim X_\lambda].$$

- (ii) For $\lambda \in \mathcal{P}(\mathbf{m}(k))$, we have

$$\text{IC}(\overline{X}_\lambda, \bar{\mathbf{Q}}_l) \simeq \text{IC}(\mathcal{X}_{\mathbf{m}(k)}, \mathcal{L}_{\rho_\lambda})|_{\mathcal{X}_{\mathbf{m},\text{uni}}} \quad (\text{up to shift}).$$

Theorem 7.13 ([S3, Cor. 7.4, Thm. 8.17]). *Assume that $\mathbf{m} \in \mathcal{Q}_{n,r}^0$. Then $(\pi_{\mathbf{m},1})_* \bar{\mathbf{Q}}_l[d'_{\mathbf{m}}]$ is a semisimple perverse sheaf on $\mathcal{X}_{\mathbf{m},\text{uni}}$ equipped with $W_{n,r}$ -action, and is decomposed as*

$$(\pi_{\mathbf{m},1})_* \bar{\mathbf{Q}}_l[d'_{\mathbf{m}}] \simeq \bigoplus_{\lambda \in \widetilde{\mathcal{P}}(\mathbf{m})} \tilde{V}_\lambda \otimes \text{IC}(\overline{X}_\lambda, \bar{\mathbf{Q}}_l)[\dim X_\lambda].$$

Remarks 7.14. (i) Our final goal is the Springer correspondence for $W_{n,r}$ given in Theorem 7.13. However, the group $W_{n,r}$ and the flag variety \mathcal{B}^θ

have no direct connection, and it is difficult to prove Theorem 7.13 directly. On the other hand, since $W_{\mathbf{m}}^{\natural}$ is a parabolic subgroup of W_n , it has a close relationship with \mathcal{B}^{θ} . Hence we first prove Theorem 7.12, and by making use of the relation (ii) in the theorem, we can prove Theorem 7.13.

(ii) Theorem 7.12 (ii) shows that the closure $\overline{X_{\lambda}}$ of X_{λ} is determined canonically from the decomposition of $(\pi_1^{(\mathbf{m})})_* \bar{\mathbf{Q}}_l$. But there is no confidential reason for the choice of X_{λ} . Our X_{λ} is just one of such choices.

7.15. We consider the Springer fibre for \mathcal{X}_{uni} . For $z = (x, \mathbf{v}) \in \mathcal{X}_{\mathbf{m}, \text{uni}}$, put

$$\begin{aligned} \mathcal{B}_z^{\theta} &= \{gB^{\theta} \in \mathcal{B}^{\theta} \mid g^{-1}xg \in B^{\iota\theta}, g^{-1}\mathbf{v} \in M_n^{r-1}\}, \\ \mathcal{B}_z^{\theta, (\mathbf{m})} &= \{gB^{\theta} \in \mathcal{B}^{\theta} \mid g^{-1}xg \in B^{\iota\theta}, g^{-1}\mathbf{v} \in \prod_{i=1}^{r-1} M_{p_i}\}. \end{aligned}$$

Hence $\mathcal{B}_z^{\theta, (\mathbf{m})} \subset \mathcal{B}_z^{\theta}$. $\mathcal{B}_z^{\theta} \simeq \pi_1^{-1}(z)$ is called the **Springer fibre** of z , and $\mathcal{B}_z^{\theta, (\mathbf{m})} \simeq (\pi_1^{(\mathbf{m})})^{-1}(z)$ is called the **small Springer fibre** of z .

Remark 7.16. Since X_{λ} is not a single H -orbit if $r \geq 3$, it is not apriori true that $\dim \mathcal{B}_z^{\theta}, \dim \mathcal{B}_z^{\theta, (\mathbf{m})}$ are constant on X_{λ} . In fact, this does not hold in general if $r \geq 3$. (But compare it with Lemma 8.9 in the enhanced case).

Assume $\mathbf{m} \in \mathcal{Q}_{n,r}^0$, and $\lambda \in \widetilde{\mathcal{P}}(\mathbf{m})$. (Then $X_{\lambda} \subset \mathcal{X}_{\mathbf{m}, \text{uni}}$ by 7.7 (iii).) Put

$$d_{\lambda} = (\dim \mathcal{X}_{\mathbf{m}, \text{uni}} - \dim X_{\lambda})/2.$$

Lemma 7.17. Assume that $\mathbf{m} \in \mathcal{Q}_{n,r}^0$ and $\lambda \in \widetilde{\mathcal{P}}(\mathbf{m})$. For any $z \in X_{\lambda}$, we have

$$\dim \mathcal{B}_z^{\theta, (\mathbf{m})} \geq d_{\lambda}.$$

The set $\{z \in X_{\lambda} \mid \dim \mathcal{B}_z^{\theta, (\mathbf{m})} = d_{\lambda}\}$ forms an open dense subset of X_{λ} .

The following result is a generalization of Corollary 5.9.

Proposition 7.18. Assume that $\mathbf{m} \in \mathcal{Q}_{n,r}^0$, and $\lambda \in \widetilde{\mathcal{P}}(\mathbf{m})$. Take $z \in X_{\lambda}$ such that $\dim \mathcal{B}_z^{\theta, (\mathbf{m})} = d_{\lambda}$. Then

- (i) $H^{2d_{\lambda}}(\mathcal{B}_z^{\theta, (\mathbf{m})}, \bar{\mathbf{Q}}_l) \simeq V_{\lambda}^{\natural}$ as $W_{\mathbf{m}}^{\natural}$ -modules.
- (ii) $H^{2d_{\lambda}}(\mathcal{B}_z^{\theta}, \bar{\mathbf{Q}}_l) \simeq \tilde{V}_{\lambda}$ as $W_{n,r}$ -modules.

In particular, the map $X_{\lambda} \mapsto H^{2d_{\lambda}}(\mathcal{B}_z^{\theta}, \bar{\mathbf{Q}}_l)$ gives a bijective correspondence

$$\{X_{\lambda} \mid \lambda \in \mathcal{P}_{n,r}\} \simeq W_{n,r}^{\wedge}.$$

§8. Enhanced variety of higher level

8.1. We follow the notation in §4. As a generalization of the enhanced variety $G \times V$, we consider $\mathcal{X} = G \times V^{r-1}$ on which G acts diagonally. $\mathcal{X} = \mathcal{X}^{\text{en}}$ is called the **enhanced variety of level r** . We also consider $\mathcal{X}_{\text{uni}} = G_{\text{uni}} \times V^{r-1}$. The discussion in the case of exotic symmetric space of higher level can be applied also to the present situation, and in fact the arguments become simpler. We fix $\mathbf{m} \in \mathcal{Q}_{n,r}$ (here we don't need to assume $\mathbf{m} \in \mathcal{Q}_{n,r}^0$), and consider the varieties

$$\begin{aligned}\widetilde{\mathcal{X}}_{\mathbf{m}} &= \{(x, \mathbf{v}, gB) \in G \times V^{r-1} \times \mathcal{B} \mid g^{-1}xg \in B, g^{-1}\mathbf{v} \in \prod_{i=1}^{r-1} M_{p_i}\}, \\ \mathcal{X}_{\mathbf{m}} &= \bigcup_{g \in G} g(B \times \prod_i M_{p_i}).\end{aligned}$$

We define a morphism $\pi^{(\mathbf{m})} : \widetilde{\mathcal{X}}_{\mathbf{m}} \rightarrow \mathcal{X}_{\mathbf{m}}$ by $(x, \mathbf{v}, gB) \mapsto (x, \mathbf{v})$.

Recall that $S_{\mathbf{m}} = S_{m_1} \times \cdots \times S_{m_r}$ is a subgroup of S_n . Put $d_{\mathbf{m}} = \dim \mathcal{X}_{\mathbf{m}}$. As a generalization of (4.7.1), we have the following result.

Theorem 8.2 ([S3, Thm. 4.5]). *For each $\mathbf{m} \in \mathcal{Q}_{n,r}$, $\pi_*^{(\mathbf{m})} \bar{\mathbf{Q}}_l[d_{\mathbf{m}}]$ is a semisimple perverse sheaf on $\mathcal{X}_{\mathbf{m}}$ equipped with $S_{\mathbf{m}}$ -action, and is decomposed as*

$$\pi_*^{(\mathbf{m})} \bar{\mathbf{Q}}_l[d_{\mathbf{m}}] \simeq \bigoplus_{\rho \in S_{\mathbf{m}}^{\wedge}} \rho \otimes \text{IC}(\mathcal{X}_{\mathbf{m}}, \mathcal{L}_{\rho})[d_{\mathbf{m}}],$$

where \mathcal{L}_{ρ} is a simple local system on a certain open dense subset of $\mathcal{X}_{\mathbf{m}}$.

8.3. Put $\mathcal{X}_{\mathbf{m},\text{uni}} = \mathcal{X}_{\mathbf{m}} \cap \mathcal{X}_{\text{uni}}$, and $\widetilde{\mathcal{X}}_{\mathbf{m},\text{uni}} = (\pi^{(\mathbf{m})})^{-1}(\mathcal{X}_{\mathbf{m},\text{uni}})$. Let $\pi_1^{(\mathbf{m})} : \widetilde{\mathcal{X}}_{\mathbf{m},\text{uni}} \rightarrow \mathcal{X}_{\mathbf{m},\text{uni}}$ be the restriction of $\pi^{(\mathbf{m})}$ on $\widetilde{\mathcal{X}}_{\mathbf{m},\text{uni}}$. We consider the complex $(\pi_1^{(\mathbf{m})})_* \bar{\mathbf{Q}}_l$. As in the exotic case, \mathcal{X}_{uni} has infinitely many G -orbits if $r \geq 3$. So, in order to describe the decomposition of $(\pi_1^{(\mathbf{m})})_* \bar{\mathbf{Q}}_l$, we need to introduce varieties X_{λ} as in the exotic case. However, in the enhanced case the situation is better than the exotic case. In fact we have a partition of \mathcal{X}_{uni} into pieces X_{λ} indexed by $\lambda \in \mathcal{P}_{n,r}$ ([S3, 5.3]),

$$(8.3.1) \quad \mathcal{X}_{\text{uni}} = \coprod_{\lambda \in \mathcal{P}_{n,r}} X_{\lambda}$$

X_{λ} is defined as follows; recall the notation in 4.5. Take $(x, \mathbf{v}) \in \mathcal{X}_{\text{uni}}$ with $\mathbf{v} = (v_1, \dots, v_{r-1})$. Put $\bar{V} = V/E^x v_1$ and $\bar{G} = GL(\bar{V})$. We consider the variety $\mathcal{X}'_{\text{uni}} = \bar{G}_{\text{uni}} \times \bar{V}^{r-2}$. Assume that $(x, v_1) \in G_{\text{uni}} \times V$ is of type $(\lambda^{(1)}, \nu')$, where $\nu = \lambda^{(1)} + \nu'$ is the Jordan type of x . Then the type of $\bar{x} = x|_{\bar{V}}$ is ν' . Put $\bar{\mathbf{v}} = (\bar{v}_2, \dots, \bar{v}_{r-1})$, where \bar{v}_i is the image of v_i on \bar{V} . Thus $(\bar{x}, \bar{\mathbf{v}}) \in \mathcal{X}'_{\text{uni}}$. By induction, we have a partition of $\mathcal{X}'_{\text{uni}}$ as in (8.3.1), and there exists a unique piece $X'_{\lambda'}$ containing $(\bar{x}, \bar{\mathbf{v}})$. If we write $\lambda' = (\lambda^{(2)}, \dots, \lambda^{(r)})$, we obtain $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)}) \in \mathcal{P}_{n,r}$. The attachment $(x, \mathbf{v}) \mapsto \lambda$, $\mathcal{X}_{\text{uni}} \rightarrow \mathcal{P}_{n,r}$ determines the set X_{λ} as the inverse image of λ . This gives the required partition.

As in the exotic case (Proposition 7.8), we have the following result.

Proposition 8.4. *X_{λ} is a locally closed, smooth, irreducible, G -stable subvariety of \mathcal{X}_{uni} satisfying the following properties. Assume that $\mathbf{m} \in \mathcal{Q}_{n,r}$.*

(i) *We have*

$$\dim X_{\lambda} = (n^2 - n - 2n(\lambda)) + \sum_{i=1}^{r-1} (r-i)|\lambda^{(i)}|.$$

(ii) $\mathcal{X}_{\mathbf{m}, \text{uni}} = \bar{X}_{\lambda(\mathbf{m})}$, (see Prop. 7.8 for the definition $\lambda(\mathbf{m})$).

(iii) For $\mu \in \mathcal{P}(\mathbf{m})$, we have $X_{\mu} \subset \mathcal{X}_{\mathbf{m}, \text{uni}}$.

(iv) If $r = 2$, X_{λ} coincides with the G -orbit \mathcal{O}_{λ} .

Remark 8.5. In general, the partition (8.3.1) is not compatible with closure relations, namely, the closure \bar{X}_{λ} is not a union of pieces X_{μ} . However, we have a somewhat weaker result ([S3, Prop. 5.11]); for each $\lambda \in \mathcal{P}_{n,r}$, we have

$$\bar{X}_{\lambda} \subset \bigcup_{\mu \leq \lambda} X_{\mu}.$$

The following result gives the Springer correspondence in the case of the enhanced variety of higher level. Note that (i) of the theorem was proved independently by Li [Li].

Theorem 8.6 ([S3, Thm. 8.13], [Li, Thm. 3.2.6]). *Assume that $\mathbf{m} \in \mathcal{Q}_{n,r}$ and put $d'_{\mathbf{m}} = \dim \mathcal{X}_{\mathbf{m}, \text{uni}}$.*

(i) $(\pi_1^{(\mathbf{m})})_* \bar{\mathbf{Q}}_l[d'_{\mathbf{m}}]$ is a semisimple perverse sheaf on $\mathcal{X}_{\mathbf{m}, \text{uni}}$ equipped with $S_{\mathbf{m}}$ -action, and is decomposed as

$$(\pi_1^{(\mathbf{m})})_* \bar{\mathbf{Q}}_l[d'_{\mathbf{m}}] \simeq \bigoplus_{\lambda \in \mathcal{P}(\mathbf{m})} \rho_\lambda \otimes \mathrm{IC}(\bar{X}_\lambda, \bar{\mathbf{Q}}_l)[\dim X_\lambda].$$

(ii) For $\lambda \in \mathcal{P}(\mathbf{m})$, we have

$$\mathrm{IC}(\bar{X}_\lambda, \bar{\mathbf{Q}}_l) \simeq \mathrm{IC}(\mathcal{X}_{\mathbf{m}}, \mathcal{L}_{\rho_\lambda})|_{\mathcal{X}_{\mathbf{m}, \mathrm{uni}}} \quad (\text{up to shift}).$$

8.7. For $z = (x, \mathbf{v}) \in \mathcal{X}_{\mathbf{m}}$, we define the Springer fibre $\mathcal{B}_z^{(\mathbf{m})} \simeq (\pi^{(\mathbf{m})})^{-1}(z)$ by

$$\mathcal{B}_z^{(\mathbf{m})} = \{gB \in \mathcal{B} \mid g^{-1}xg \in B, g^{-1}\mathbf{v} \in \prod_{i=1}^{r-1} M_{p_i}\}.$$

As in the exotic case, for $\lambda \in \mathcal{P}(\mathbf{m})$ with $\mathbf{m} \in \mathcal{Q}_{n,r}$, we define d_λ by $d_\lambda = (\dim \mathcal{X}_{\mathbf{m}, \mathrm{uni}} - \dim X_\lambda)/2$. Then one can check easily that $d_\lambda = n(\lambda)$. By a similar argument as in the proof of Lemma 7.17, we see that $\dim \mathcal{B}_z^{(\mathbf{m})} \geq d_\lambda$. On the other hand, since $\mathcal{B}_z^{(\mathbf{m})} \subset \mathcal{B}_x$, and since it is well-known that $\dim \mathcal{B}_x = n(\lambda)$, we have $\dim \mathcal{B}_z^{(\mathbf{m})} \leq n(\lambda)$. It follows that

Lemma 8.8. *Assume that $\lambda \in \mathcal{P}(\mathbf{m})$. Then for any $z \in X_\lambda$, we have $\dim \mathcal{B}_z^{(\mathbf{m})} = d_\lambda = n(\lambda)$.*

The following result is an analogue of Proposition 7.18, and was proved in [S3, Prop. 8.16]. Note that a similar result was proved by [Li, Cor. 3.2.9] for the Borel-Moore homology.

Proposition 8.9. *Assume that $\mathbf{m} \in \mathcal{Q}_{n,r}$, and $\lambda \in \mathcal{P}(\mathbf{m})$. Then for any $z \in X_\lambda$, we have $H^{2d_\lambda}(\mathcal{B}_z^{(\mathbf{m})}, \bar{\mathbf{Q}}_l) \simeq \rho_\lambda$ as $S_{\mathbf{m}}$ -modules.*

8.10. We consider a generalization of the discussion on Kostka polynomials given in 4.10. For a general r , Kostka functions are given as rational functions $K_{\lambda, \mu}^\pm(t)$, indexed by $\lambda, \mu \in \mathcal{P}_{n,r}$ and by a sign $+$ or $-$. They are defined by fixing a total order on $\mathcal{P}_{n,r}$ compatible with the partial order \leq on $\mathcal{P}_{n,r}$. See [S1, 2, 4, 5] for details. The modified Kostka function is defined, as in the case $r = 2$, by $\tilde{K}_{\lambda, \mu}^\pm(t) = t^{a(\mu)} K_{\lambda, \mu}^\pm(t^{-1})$. The connection of those Kostka functions and the enhanced variety of higher level was discussed in [S4]. (Actually only the functions $K_{\lambda, \mu}^-(t)$ labelled by negative sign $-$ behave well.) Here we concentrate ourselves to the special case where $\mu = (-, \dots, -, \mu^{(r-1)}, \mu^{(r)})$. Note in that case, X_μ consists of a single G -orbit. The following (partial) result can be compared to Theorem 4.11 in the $r = 2$ case.

Proposition 8.11 ([S4, Prop. 6.8]). *Take $\lambda, \mu \in \mathcal{P}_{n,r}$, and assume that $\mu = (-, \dots, -, \mu^{(r-1)}, \mu^{(r)})$. Consider $K = \text{IC}(\overline{X}_\lambda, \bar{Q}_l)$ on \mathcal{X}_{uni} . Then*

- (i) $\mathcal{H}_z^i K = 0$ for $z \in X_\mu$ if i is odd.
- (ii) Assume that $\mu \leq \lambda$. Then for any $z \in X_\mu \subset \overline{X}_\lambda$, we have

$$(8.11.1) \quad \tilde{K}_{\lambda, \mu}^-(t) = t^{a(\lambda)} \sum_{i \geq 0} (\dim \mathcal{H}_z^{2i} K) t^{ri}.$$

Note that by (8.11.1), in this special case, $\tilde{K}_{\lambda, \mu}^-(t)$ and so $K_{\lambda, \mu}^-(t)$ turns out to be polynomials in $\mathbf{Z}[t]$, which are independent of the choice of the total order.

8.12. An analogue of the theorem of Lascoux-Schützenberger (3.4.1) to the Kostka functions associated to r -partitions was discussed in [LS], [S5]. Let $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)}) \in \mathcal{P}_{n,r}$. An r -tuple $T = (T^{(1)}, \dots, T^{(r)})$ of tableaux is called a semistandard tableau of shape λ if $T^{(i)}$ is a semistandard tableau of shape $\lambda^{(i)}$ with letters in $\{1, \dots, n\}$. The weight $\xi = (\xi_1, \dots, \xi_n)$ of T is defined by putting ξ_i the number of letters i contained in the boxes in $\coprod_j T^{(j)}$ for each i . For $\lambda \in \mathcal{P}_{n,r}$ and $\xi \in \mathcal{P}_n$, we denote by $SST(\lambda, \xi)$ the set of semistandard tableaux of shape λ and weight ξ . Put for $\lambda \in \mathcal{P}_{n,r}$,

$$b(\lambda) = a(\lambda) - r \cdot n(\lambda) = |\lambda^{(2)}| + 2|\lambda^{(3)}| + \dots + (r-1)|\lambda^{(r)}|.$$

For $T \in SST(\lambda, \xi)$, the charge $c(T)$ is defined in [LS] (in the case $r = 2$), [S5] (for general r). Then we have the following result.

Theorem 8.13 ([LS, Thm. 3.12], [S5, Thm. 3.14]). *Let $\lambda, \mu \in \mathcal{P}_{n,r}$, and assume that $\mu = (-, \dots, -, \xi)$ with $\xi \in \mathcal{P}_n$. Then*

$$K_{\lambda, \mu}^-(t) = t^{b(\mu) - b(\lambda)} \sum_{T \in SST(\lambda, \xi)} t^{r \cdot c(T)}.$$

REFERENCES

- [AH] P.N. Achar and A. Henderson; Orbit closures in the enhanced nilpotent cone, Adv. in Math. **219** (2008), no. 1, 27-62, Corrigendum, ibid. **228** (2011), 2984-2988.
- [BM] W. Borho and R. MacPherson; Representations des groupes de Weyl et homologie d'intersection pour les varietes nilpotentes, C.R. Acad. Sci., Paris, Sr.I **292** (1981), 707-710.
- [BD] M. Boyarchenko and V. Drinfeld; Character sheaves on unipotent groups in positive characteristic: Foundations, October 2008, Selecta Math. DOI:10.1007/s00029-013-0133-7, arXiv:0810.0794v1.

- [DP] C. De Concini and C. Procesi, Symmetric functions, conjugacy classes, and the flag variety, *Invent. Math.* **64** (1981), 203-230.
- [FG] M. Finkelberg and V. Ginzburg; Cherednik algebras for algebraic curves, *Progress in Math.* **284**, (2010), 121-153.
- [FGT] M. Finkelberg, V. Ginzburg and R. Travkin; Mirabolic affine Grassmannian and character sheaves, *Selecta Math.* **14** (2009), 607-628.
- [Gi] V. Ginzburg; Admissible modules on a symmetric space, *Astérisque* **173-174** (1989), 199-255.
- [Gr] I. Grojnowski; Character sheaves on symmetric spaces, Ph.D. thesis, MIT 1992.
- [H] A. Henderson; Fourier transform, parabolic induction, and nilpotent orbits, *Transformation groups*, **6**, (2001), 353-370.
- [K1] S. Kato; An exotic Deligne-Langlands correspondence for symplectic groups, *Duke Math. J.* **148** (2009), 306-371.
- [K2] S. Kato; Deformations of nilpotent cones and Springer correspondence, *Amer. J. of Math.* **133** (2011), 519-553.
- [K3] S. Kato; An algebraic study of extension algebras, preprint. arXiv:1207.4640.
- [Li] Y. Li; A class of perverse sheaves on framed representation varieties of the Jordan quiver, *J. Algebra* **386** (2013), 113-130.
- [L1] G. Lusztig; Green polynomials and singularities of unipotent classes, *Adv. in Math.* **42** (1981), 169-178.
- [L2] G. Lusztig; Intersection cohomology complexes on a reductive group, *Invent. Math.* **75** (1984), 205-272.
- [L3] G. Lusztig; Character sheaves, I \sim V, *Adv. in Math.* **56** (1983), 193-237, **57** (1985), 226-265, **57** (1985), 266-315, **59** (1986), 1-63, **61** (1986), 103-155.
- [L4] G. Lusztig; Character sheaves on disconnected groups, I \sim X, *Represent. Theory* **7**(2003), 374-403, Errata **8**(2004), 179-179, **8**(2004), 72-124, **8**(2004), 125-144, **8**(2004), 145-178, **8**(2004), 346-376, **8**(2004), 377-413, **9**(2005), 209-266, **10**(2006), 314-352, **10**(2006), 353-379, **13**(2009), 82-140.
- [LS] S. Liu and T. Shoji; Double Kostka polynomials and Hall bimodule, preprint. arXiv: 1501.05996.
- [M] I.G. Macdonald; "Symmetric functions and Hall polynomials", Clarendon Press, Oxford, 1995.
- [S1] T. Shoji; Green functions associated to complex reflection groups, *J. Algebra* **245** (2001), 650-694.
- [S2] T. Shoji; Green functions attached to limit symbols, in "Representation theory of algebraic groups and quantum groups", *Adv. Stud. Pure Math.*, **40**. Math. Soc. Japan, Tokyo, 2004, 443-467.
- [S3] T. Shoji; Exotic symmetric spaces of higher level - Springer correspondence for complex reflection groups, to appear in *Transform. Groups*.
- [S4] T. Shoji; Enhanced variety of higher level and Kostka functions associated to complex reflection groups, preprint. arXiv: 1507.01240.
- [S5] T. Shoji; Kostka functions associated to complex reflection groups, preprint.
- [Sp] T.A. Springer; Trigonometric sums, Green functions of finite groups and representation theory of Weyl groups, *Invent. Math.* **36** (1976), 173-207.
- [SS1] T. Shoji and K. Sorlin; Exotic symmetric space over a finite field, I, *Transformation Groups*, **18**, (2013) 877-929.

- [SS2] T. Shoji and K. Sorlin; Exotic symmetric space over a finite field, II, Transformation Groups **19** (2014), 887-926.
- [SS3] T. Shoji and K. Sorlin; Exotic symmetric space over a finite field, III. Transformation Groups **19** (2014), 1149-1198.
- [T] R. Travkin; Mirabolic Robinson-Schensted-Knuth correspondence, Selecta Mathematica (New series) **14** (2009), 727-758.

T. Shoji
Department of Mathematics, Tongji University
1239 Siping Road, Shanghai 200092, P. R. China
E-mail: shoji@tongji.edu.cn